Off-diagonal Bethe ansatz for exactly solvable models

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The Hamiltonian of the closed XXZ chain is

$$H = -\frac{1}{2}\sum_{k=1}^{N}\left(\sigma_k^x\,\sigma_{k+1}^x + \sigma_k^y\,\sigma_{k+1}^y + \cosh\eta\,\sigma_k^z\,\sigma_{k+1}^z\right),$$

where

$$\sigma_{N+1}^{\alpha} = \sigma_1^{\alpha}, \quad \alpha = x,\, y,\, z.$$

The system is **integrable**, i.e., there exist enough conserved charges

$$i\hbar\frac{\partial}{\partial t}h_i = [H, h_i] = 0, \qquad i = 1, \ldots .$$

and

$$[h_i, h_j] = 0.$$

It is convenient to introduce a generation function of these charges, the so-called transfer matrix

$$t(u) = \sum_{i=0} h_i u^i.$$

Then

$$[t(u),\ t(v)] = 0, \qquad H \propto \frac{\partial}{\partial u} \ln t(u)|_{u=0} + const,$$

or

$$H \propto h_0^{-1} h_1 + const,$$
$$h_0\, \sigma_i^{\alpha}\, h_0^{-1} = \sigma_{i+1}^{\alpha}.$$

The eigenstates and the corresponding eigenvalues can be obtained by Quantum Inverse Scattering Method (QISM). In the framework of QISM, the monodromy matrix $T(u)$

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},$$

has played a central role. It is built from the six-vertex R-matrix of

$$T_0(u) = R_{0N}(u-\theta_N)\ldots R_{01}(u-\theta_1),$$

where the well-known six-vertex R-matrix is given by

$$R(u) = \frac{1}{\sinh\eta}\begin{pmatrix} \sinh(u+\eta) & & & \\ & \sinh u & \sin\eta & \\ & \sinh\eta & \sin u & \\ & & & \sinh(u+\eta) \end{pmatrix}.$$

The transfer matrix is $t(u) = trT(u) = A(u) + D(u)$.

In the case of N=1,

$$A(u) = \frac{1}{\sinh\eta}\begin{pmatrix} \sinh(u-\theta_1+\eta) & \\ & \sinh(u-\theta_1) \end{pmatrix}, \quad B(u) = \begin{pmatrix} & \\ 1 & \end{pmatrix},$$

$$C(u) = \begin{pmatrix} & 1 \\ & \end{pmatrix}, \quad D(u) = \frac{1}{\sinh\eta}\begin{pmatrix} \sinh(u-\theta_1) & \\ & \sinh(u-\theta_1+\eta) \end{pmatrix}.$$

In the case of N=2

$$A(u) = A_2(u)A_1(u) + B_2(u)C_1(u), \quad B(u) = A_2(u)B_1(u) + B_2(u)D_1(u),$$
$$C(u) = C_2(u)A_1(u) + D_2(u)C_1(u), \quad D(u) = C_2(u)B_1(u) + D_2(u)D_1(u).$$

$$\vdots$$

The R-matrix satisfies the Yang-Baxter equation (YBE)

$$R_{12}(u-v)\,R_{13}(u)\,R_{23}(v)=R_{23}(v)\,R_{13}(u)\,R_{12}(u-v). \tag{1}$$

The above fundamental relation leads to the following so-called RLL relation between the monodromy matrix

$$R_{0\,0'}(u-v)\,T_0(u)\,T_{0'}(v)=T_{0'}(v)\,T_0(u)\,R_{0\,0'}(u-v). \tag{2}$$

This leads to $[t(u),t(v)]=0$.

In terms of the matrix elements of the monodromy matrix, the RLL relation read

$$B(u)B(v) = B(v)B(u), \tag{3}$$

$$A(u)B(v) = \frac{\sinh(u-v-\eta)}{\sinh(u-v)}B(v)A(u) + \frac{\eta}{\sinh(u-v)}B(u)A(v), \tag{4}$$

$$D(u)B(v) = \frac{\sinh(u-v+\eta)}{\sinh(u-v)}B(v)D(u) - \frac{\eta}{\sinh(u-v)}B(u)D(v), \tag{5}$$

$$\vdots$$

There exists a quasi-vacuum state (or reference state) $|\Omega\rangle$ such that

$$|\Omega\rangle = |\uparrow,\ldots,\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{6}$$

$$A(u)|\Omega\rangle = a(u)|\Omega\rangle = \prod_{j=1}^{N}(\sinh u - \theta_j + \eta)|\Omega\rangle, \tag{7}$$

$$D(u)|\Omega\rangle = d(u)|\Omega\rangle = \prod_{j=1}^{N}\sinh(u-\theta_j)|\Omega\rangle, \tag{8}$$

$$C(u)|\Omega\rangle = 0, \quad B(u)|\Omega\rangle \neq 0. \tag{9}$$

Let us introduce the Bethe state

$$|\lambda_1, \ldots, \lambda_M\rangle = B(\lambda_1) \ldots B(\lambda_M) \, |\Omega\rangle. \qquad (10)$$

The action of the transfer matrix reads

$$\begin{aligned} t(u)|\lambda_1, \ldots, \lambda_M\rangle &= \prod_{i=1}^{M} \frac{\sinh(u - \lambda_i - \eta)}{\sinh(u - \lambda_i)} a(u)|\lambda_1, \ldots, \lambda_M\rangle \\ &+ \prod_{i=1}^{M} \frac{\sinh(u - \lambda_i + \eta)}{\sinh(u - \lambda_i)} d(u)|\lambda_1, \ldots, \lambda_M\rangle \\ &+ \text{unwanted terms.} \end{aligned}$$

If the parameters $\{\lambda_i\}$ needs satisfy Bethe ansatz equations,

$$\prod_{k\neq j}^{M}\frac{\sinh(\lambda_j-\lambda_k+\eta)}{\sinh(\lambda_j-\lambda_k-\eta)}=\prod_{l=1}^{N}\frac{\sinh(\lambda_j-\theta_l+\eta)}{\sinh(\lambda_j-\theta_l)},\qquad j=1,\ldots,M. \tag{11}$$

Then the Bethe states become the common eigenstates of $t(u)$ with eigenvalue $\Lambda(u)$

$$t(u)|\lambda_1,\ldots,\lambda_M\rangle=\Lambda(u)|\lambda_1,\ldots,\lambda_M\rangle,$$

where $\Lambda(u)=\Lambda(u;\lambda_1,\ldots,\lambda_M)$ is given by

$$\begin{aligned}\Lambda(u) &= a(u)\prod_{i=1}^{M}\frac{\sinh(u-\lambda_i-\eta)}{\sinh(u-\lambda_i)}+d(u)\prod_{i=1}^{M}\frac{\sinh(u-\lambda_i+\eta)}{\sinh(u-\lambda_i)},\\ &= a(u)\frac{Q(u-\eta)}{Q(u)}+d(u)\frac{Q(u+\eta)}{Q(u)},\end{aligned} \tag{12}$$

where

$$Q(u)=\prod_{i=1}^{M}\sinh(u-\lambda_i). \tag{13}$$

The Hamiltonian of the XXZ chain with twisted boundary condition is

$$H = -\frac{1}{2}\sum_{k=1}^{N}\left(\sigma_k^x\,\sigma_{k+1}^x + \sigma_k^y\,\sigma_{k+1}^y + \cosh\eta\,\sigma_k^z\,\sigma_{k+1}^z\right),$$

where

$$\sigma_{N+1}^{\alpha} = e^{i\phi\sigma_1^z}\,\sigma_1^{\alpha}\,e^{-i\phi\sigma_1^z}, \quad \alpha = x, y, z.$$

The phase factor ϕ can be arbitrary complex number. The system is **integrable**, i.e., the corresponding transfer matrix can be constructed as

$$t(u) = tr(e^{i\phi\sigma^z}\,T(u)) = e^{i\phi}A(u) + e^{-i\phi}D(u).$$

The transfer matrix can diagonalized by algebraic Bethe ansatz similar as that of periodic case. The Bethe state is the same as (10), namely,

$$|\lambda_1,\ldots,\lambda_M\rangle = B(\lambda_1)\ldots B(\lambda_M)\,|\Omega\rangle.$$

If the parameters $\{\lambda_i\}$ satisfies Bethe ansatz equations,

$$\prod_{k\neq j}^{M}\frac{\sinh(\lambda_j-\lambda_k+\eta)}{\sinh(\lambda_j-\lambda_k-\eta)}=e^{2i\phi}\prod_{l=1}^{N}\frac{\sinh(\lambda_j-\theta_l+\eta)}{\sinh(\lambda_j-\theta_l)},\qquad j=1,\ldots,M. \tag{14}$$

Then the Bethe states become the common eigenstates of $t(u)$ with eigenvalue $\Lambda(u)$

$$\begin{aligned}\Lambda(u) &= e^{i\phi}a(u)\prod_{i=1}^{M}\frac{\sinh(u-\lambda_i-\eta)}{\sinh(u-\lambda_i)}+e^{-i\phi}d(u)\prod_{i=1}^{M}\frac{\sinh(u-\lambda_i+\eta)}{\sinh(u-\lambda_i)},\\ &= e^{i\phi}a(u)\frac{Q(u-\eta)}{Q(u)}+e^{-i\phi}d(u)\frac{Q(u+\eta)}{Q(\eta)}.\end{aligned} \tag{15}$$

The Hamiltonian of the XXZ chain with antiperiodic boundary condition is

$$H = -\frac{1}{2}\sum_{k=1}^{N}\left(\sigma_k^x\,\sigma_{k+1}^x + \sigma_k^y\,\sigma_{k+1}^y + \cosh\eta\,\sigma_k^z\,\sigma_{k+1}^z\right),$$

where

$$\sigma_{N+1}^{\alpha} = \sigma_1^x\,\sigma_1^{\alpha}\,\sigma_1^x, \quad \alpha = x,\, y,\, z.$$

The system is **integrable**, i.e., the corresponding transfer matrix can be constructed as

$$t(u) = tr(\sigma^x T(u)) = B(u) + C(u).$$

The model is a typical integrable without $U(1)$ symmetry. Most of conventional Bethe ansatz method fails to give the solution because of the lack of a proper vacuum (or reference) state.

Recently, we give a solution to the spectrum problem of the corresponding transfer matrix in

- *Phys. Rev. Lett.* **111** (2013), 137201.

Let $|\Psi\rangle$ be an eigenstate of the transfer matrix with an eigenvalue

$$t(u)|\Psi\rangle = \Lambda(u)|\Psi\rangle.$$

Due to the fact that $|\Psi\rangle$ does not depend on u, we can derive the following properties which can determine $\Lambda(u)$ completely

$$\Lambda(u), \text{ as a function of } u, \text{ is a trigonometric polynomial of degree } N-1, \quad (16)$$

$$\Lambda(u+i\pi) = (-1)^{N-1}\Lambda(u), \quad (17)$$

$$\Lambda(\theta_j)\,\Lambda(\theta_j-\eta) = -a(\theta_j)d(\theta_j-\eta), \quad j=1,\ldots,N. \quad (18)$$

The solution to the above equations is given by

$$\Lambda(u) = e^{u} a(u) \frac{Q(u-\eta)}{Q(u)} - e^{-u-\eta} d(u) \frac{Q(u+\eta)}{Q(u)} - c(u) \frac{a(u)d(u)}{Q(u)}, \quad (19)$$

$$Q(u) = \prod_{j=1}^{N} \sinh(u - \lambda_j),$$

$$c(u) = e^{u - N\eta + \sum_{l=1}^{N}(\theta_l - \lambda_l)} - e^{-u-\eta-\sum_{l=1}^{N}(\theta_l - \lambda_l)}.$$

The parameters $\{\lambda_j\}$ satisfy the associated Bethe ansatz equations

$$e^{\lambda_j} a(\lambda_j) Q(\lambda_j - \eta) - e^{-\lambda_j - \eta} d(\lambda_j) Q(\lambda_j + \eta) - c(\lambda_j) a(\lambda_j) d(\lambda_j) = 0,$$
$$j = 1, \ldots, N. \quad (20)$$

- *JSTAT* (2015), **P05014**.

The eigenstate of the transfer matrix can be given by the following Bethe state

$$|\lambda_1, \ldots, \lambda_N\rangle = \prod_{j=1}^{N} \frac{D(\lambda_j)}{d(\lambda_j)} |\Omega; \{\theta_j\}\rangle, \tag{21}$$

where the parameters $\{\lambda_j | j = 1, \ldots, N\}$ satisfy the BAEs (20). The "reference state" $|\Omega; \{\theta_j\}\rangle$ is given by the following spin coherent state

$$|\Omega; \{\theta_j\}\rangle = \sum_{l=0}^{\infty} \frac{(B^-)^l}{[l]_q!} |0\rangle = \sum_{l=0}^{N} \frac{(B^-)^l}{[l]_q!} |0\rangle, \tag{22}$$

where the q-integers $\{[l]_q | l = 0, \cdots\}$ and the operator B^- are given by

$$[l]_q = \frac{1 - q^{2l}}{1 - q^2}, \quad [0]_q = 1,$$
$$[l]_q! = [l]_q \, [l-1]_q \cdots [1]_q, \quad q = e^{\eta}, \tag{23}$$

$$B^- = \sum_{l=1}^{N} e^{\theta_l + \frac{(N-1)\eta}{2}} \, e^{\frac{\eta}{2} \sum_{k=l+1}^{N} \sigma_k^z} \, \sigma_l^- \, e^{-\frac{\eta}{2} \sum_{k=1}^{l-1} \sigma_k^z}. \tag{24}$$

Open XXZ chain Hamiltonian

$$H = -\frac{1}{2}\sum_{k=1}^{N-1}\left(\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y + \cosh\eta\sigma_k^z\sigma_{k+1}^z\right) + f_1^x\sigma_1^x + f_1^y\sigma_1^y + f_1^z\sigma_1^z + f_N^x\sigma_N^x + f_N^y\sigma_N^y + f_N^z\sigma_N^z$$

The model is **integrable**. If the components of boundary fields are parameterized by

$$F_1 = (f_1^x, f_1^y, f_1^z) = \frac{\sinh\eta}{\sinh\alpha_-\cosh\beta_-}(\coth\alpha_-\sinh\beta_-, \cosh\theta_-, i\sinh\theta_-)$$

$$F_N = (f_N^x, f_N^y, f_N^z) = \frac{\sinh\eta}{\sinh\alpha_+\cosh\beta_+}(-\coth\alpha_+\sinh\beta_+, \cosh\theta_+, i\sinh\theta_+).$$

The corresponding transfer matrix $t(u)$ can be constructed by the six-vertex R-matrix and the associated K-matrices, i.e.,

$$t(u) = tr(\mathbb{T}(u)) = tr\left(K^+(u)T(u)K^-(u)T^{-1}(-u)\right),$$

where the K-matrices $K^{\pm}(u)$ are the most general solutions of the reflection equation and its dual.

The K-matrix $K^-(u)$ is given by

$$\begin{aligned} K^-(u) &= \begin{pmatrix} K^-_{11}(u) & K^-_{12}(u) \\ K^-_{21}(u) & K^-_{22}(u) \end{pmatrix}, \\ K^-_{11}(u) &= 2\left(\sinh(\alpha_-)\cosh(\beta_-)\cosh(u) + \cosh(\alpha_-)\sinh(\beta_-)\sinh(u)\right), \\ K^-_{22}(u) &= 2\left(\sinh(\alpha_-)\cosh(\beta_-)\cosh(u) - \cosh(\alpha_-)\sinh(\beta_-)\sinh(u)\right), \\ K^-_{12}(u) &= e^{\theta_-}\sinh(2u), \quad K^-_{21}(u) = e^{-\theta_-}\sinh(2u), \end{aligned} \tag{25}$$

and it satisfies the reflection equation (RE)

$$\begin{aligned} & R_{12}(u_1-u_2)K^-_1(u_1)R_{21}(u_1+u_2)K^-_2(u_2) \\ & \quad = K^-_2(u_2)R_{12}(u_1+u_2)K^-_1(u_1)R_{21}(u_1-u_2). \end{aligned} \tag{26}$$

The dual K-matrix $K^+(u)$ satisfies the following dual RE

$$R_{12}(u_2-u_1)K_1^+(u_1)R_{21}(-u_1-u_2-2)K_2^+(u_2) \\ = K_2^+(u_2)R_{12}(-u_1-u_2-2)K_1^+(u_1)R_{21}(u_2-u_1). \quad (27)$$

The most general solution to the DRE is

$$K^+(u) = K^-(-u-\eta)\big|_{(\alpha_-,\beta_-,\theta_-)\to(-\alpha_+,-\beta_+,\theta_+)}. \quad (28)$$

The Hamiltonian can be expressed in terms of the transfer matrix

$$H = \sinh\eta \frac{\partial \ln t(u)}{\partial u}|_{u=0,\theta_j=0} - N\cosh\eta - \tanh\eta\sinh\eta.$$

For the very special case of $F_1 = (f_1^x, f_1^y, f_1^z) = (0, 0, f_1^z)$ and $F_N = (0, 0, f_N^z)$, namely,

$$H = -\frac{1}{2}\sum_{k=1}^{N-1}\left(\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y + \cosh\eta\sigma_k^z\sigma_{k+1}^z\right) + f_1^z\sigma_1^z + f_N^z\sigma_N^z$$

the model was solved by Sklyanin (J. Phys. A 21 (1988), 2375). The boundary QISM has failed to solve the spectral problem of the general case for many years. However, it can be solved by a generalized boundary QISM developed (Fan et al Nucl. Phys. B 478 (1996), 723, Cao et al Nucl. Phys. B 663 (2003), 487) for some case. In these cases, a local vacuum state does exist and the corresponding Bethe sates have similar structure as that of closed but with a different quasi-particle creation operator $\mathcal{B}(u)$ and reference state $\tilde{\Omega}$. The corresponding Bethe states are

$$|\lambda_1, \ldots, \lambda_M\rangle = \mathcal{B}(\lambda_1)\ldots\mathcal{B}(\lambda_M)\,|\tilde{\Omega}\rangle,$$

where the parameters $\{\lambda_i\}$ needs satisfy the associate Bethe ansatz equations.

For the generic $F_1 = (f_1^x, f_1^y, f_1^z)$, $F_N = (f_N^x, f_N^y, f_N^z)$ and generic anisotropic parameter Δ , the model has not been solved since Sklyanin's work in 1988 until our recent works:

- *Nucl. Phys.* **B 875** (2013), 152-165;
- *Nucl. Phys.* **B 877** (2013), 152-175.

In the solutions, we can give the eigenvalues in terms of some parameters which satisfy associated Bethe ansatz equations, without the explicit expressions of the corresponding eigenstates.

Besides the quantum Yang-Baxter equation, the R-matrix satisfies

$$\text{Initial condition}: \ R_{12}(0) = P_{12}, \tag{29}$$

$$\text{Unitarity relation}: \ R_{12}(u)R_{21}(-u) = -\frac{\sinh(u+\eta)\sinh(u-\eta)}{\sinh\eta\sinh\eta} \times \mathrm{id}, \tag{30}$$

$$\text{Crossing relation}: \ R_{12}(u) = V_1 R_{12}^{t_2}(-u-\eta)V_1, \quad V = -i\sigma^y, \tag{31}$$

$$\text{PT-symmetry}: \ R_{12}(u) = R_{21}(u) = R_{12}^{t_1\,t_2}(u), \tag{32}$$

$$Z_2\text{-symmetry}: \quad \sigma_1^i\sigma_2^i R_{1,2}(u) = R_{1,2}(u)\sigma_1^i\sigma_2^i, \quad \text{for } i = x, y, z, \tag{33}$$

$$\text{Antisymmetry}: \ R_{12}(-\eta) = -\eta(1-P) = -2\eta P^{(-)}. \tag{34}$$

In addition to reflection equations, the K-matric satisfies

$$K^-(0) = \frac{1}{2}tr(K^-(0)) \times \mathrm{id}, \quad K^-(\frac{i\pi}{2}) = \frac{1}{2}tr(K^-(\frac{i\pi}{2})) \times \sigma^z.$$

These properties and the quasi-periodic properties of R-matrix and K-matrices imply

$$\begin{aligned} t(-u-\eta) &= t(u), \quad t(u+i\pi)=t(u), \\ t(0) &= -2^3 \sinh\alpha_- \cosh\beta_- \sinh\alpha_+ \cosh\beta_+ \cosh\eta \\ &\quad \times \prod_{l=1}^{N} \frac{\sinh(\eta-\theta_l)\,\sinh(\eta+\theta_l)}{\sinh\eta\,\sinh\eta} \times \mathrm{id}, \\ t(\frac{i\pi}{2}) &= -2^3 \cosh\alpha_- \sinh\beta_- \cosh\alpha_+ \sinh\beta_+ \cosh\eta \\ &\quad \times \prod_{l=1}^{N} \frac{\sinh(\frac{i\pi}{2}+\theta_l+\eta)\sinh(\frac{i\pi}{2}+\theta_l-\eta)}{\sinh\eta\,\sinh\eta} \times \mathrm{id}, \\ \lim_{u\to\pm\infty} t(u) &= -\frac{\cosh(\theta_- - \theta_+)e^{\pm[(2N+4)u+(N+2)\eta]}}{2^{2N+1}\sinh^{2N}\eta} \times \mathrm{id} + \ldots, \end{aligned}$$

and the very operator identity

$$t(\theta_j)t(\theta_j-\eta) = -\frac{\sinh^2\eta\,\Delta_q^{(o)}(\theta_j)}{\sinh(2\theta_j+\eta)\sinh(2\theta_j-\eta)}, \quad \Delta_q^{(o)}(u) = \delta(u)\times \mathrm{id}.$$

Let $|\Psi\rangle$ be a common eigenstate of the transfer matrix with an eigenvalue $\Lambda(u)$, then

$$\Lambda(-u-\eta) = \Lambda(u), \quad \Lambda(u+i\pi) = \Lambda(u), \tag{35}$$

$$\Lambda(0) = -2^3 \sinh\alpha_- \cosh\beta_- \sinh\alpha_+ \cosh\beta_+ \cosh\eta \tag{36}$$

$$\times \prod_{l=1}^{N} \frac{\sinh(\eta-\theta_l)\,\sinh(\eta+\theta_l)}{\sinh\eta\,\sinh\eta}, \tag{37}$$

$$\Lambda(\frac{i\pi}{2}) = -2^3 \cosh\alpha_- \sinh\beta_- \cosh\alpha_+ \sinh\beta_+ \cosh\eta \tag{38}$$

$$\times \prod_{l=1}^{N} \frac{\sinh(\frac{i\pi}{2}+\theta_l+\eta)\sinh(\frac{i\pi}{2}+\theta_l-\eta)}{\sinh\eta\,\sinh\eta}, \tag{39}$$

$$\lim_{u\to\pm\infty}\Lambda(u) = -\frac{\cosh(\theta_- - \theta_+)e^{\pm[(2N+4)u+(N+2)\eta]}}{2^{2N+1}\sinh^{2N}\eta} + \ldots, \tag{40}$$

$$\Lambda(\theta_j)\Lambda(\theta_j-\eta) = -\frac{\sinh^2\eta\,\delta(\theta_j)}{\sinh(2\theta_j+\eta)\sinh(2\theta_j-\eta)}. \tag{41}$$

The function $\delta(u)$ is given by

$$\delta(u) = -2^4 \frac{\sinh(2u-2\eta)\sinh(2u+2\eta)}{\sinh\eta\,\sinh\eta} \sinh(u+\alpha_-)\sinh(u-\alpha_-)\cosh(u+\beta_-)$$
$$\times\cosh(u-\beta_-)\sinh(u+\alpha_+)\sinh(u-\alpha_+)\cosh(u+\beta_+)\cosh(u-\beta_+)$$
$$\times\prod_{l=1}^{N}\frac{\sinh(u+\theta_l+\eta)\sinh(u-\theta_l+\eta)\sinh(u+\theta_l-\eta)\sinh(u-\theta_l-\eta)}{\sinh(\eta)\sinh(\eta)\sinh(\eta)\sinh(\eta)}.$$

Moreover, it follows that $\Lambda(u)$, as an entire function of u, is a trigonometric polynomial of degree $2N+4$. Hence (35)-(41) completely determine the function $\Lambda(u)$. For this purpose, let us introduce the following functions:

$$A(u) = \prod_{l=1}^{N}\frac{\sinh(u-\theta_l+\eta)\,\sinh(u+\theta_l+\eta)}{\sinh\eta\,\sinh\eta},$$
$$a(u) = -2^2\frac{\sinh(2u+2\eta)}{\sinh(2u+\eta)}\sinh(u-\alpha_-)\cosh(u-\beta_-)$$
$$\times\sinh(u-\alpha_+)\cosh(u-\beta_+)A(u),$$
$$d(u) = a(-u-\eta).$$

$$\Lambda(u) = a(u)\frac{Q(u-\eta)}{Q(u)} + d(u)\frac{Q(u+\eta)}{Q(u)} + \frac{2c\sinh(2u)\sinh(2u+2\eta)}{Q(u)}A(u)A(-u-\eta), \quad (42)$$

where the functions $Q(u)$ is some trigonometric polynomial

$$Q(u) = \prod_{j=1}^{N}\frac{\sinh(u-\lambda_j)\sinh(u+\lambda_j+\eta)}{\sinh\eta\sinh\eta}. \quad (43)$$

the constant c is determined by the boundary parameters

$$c = \cosh((N+1)\eta+\alpha_-+\beta_-+\alpha_++\beta_+) - \cosh(\theta_--\theta_+). \quad (44)$$

The N parameters $\{\lambda_j\}$ satisfy the associated Bethe ansatz equations

$$a(\lambda_j)Q(\lambda_j-\eta) + d(\lambda_j)Q(\lambda_j+\eta) + 2c\sinh(2\lambda_j)\sinh(2\lambda_j+2\eta)A(\lambda_j)A(-\lambda_j-\eta) = 0, \quad j=1,\ldots,N, \quad (45)$$

and with the following selection rule for the roots of the above equations

$$\lambda_j \neq \lambda_l \quad \text{and} \quad \lambda_j \neq -\lambda_l - \eta.$$

- *Nucl. Phys.* **B 893** (2015), 70-88.

The associated Bethe-type eigenstates are given by

$$|\lambda_1, \cdots, \lambda_N\rangle = \mathcal{C}_{m^{(l)}}(\lambda_1|\alpha^{(l)})\,\mathcal{C}_{m^{(l)}+2}(\lambda_2|\alpha^{(l)})\cdots\mathcal{C}_{m^{(l)}+2(N-1)}(\lambda_N|\alpha^{(l)})\,|\tilde{\Omega}\rangle, \qquad (46)$$

where the two parameters $\alpha^{(l)}$ and $m^{(l)}$ are determined by the boundary parameters of K^+-matrix, while the reference state $|\tilde{\Omega}\rangle$ is determined by those of K^--matrix. The N parameters $\{\lambda_j|j = 1, \cdots, N\}$ satisfy the BAEs (45).

- QYBE& REs $\Rightarrow$ **Integrability** $\Leftrightarrow$ **Transfer matrix**

- *Intrinsic Prop.* of R(or K)-matrix $\Rightarrow$ **Operator Id.** $\Leftrightarrow$ **Solvability**

Moreover, almost all of standard R-matrices and K-matrices have such intrinsic properties.

The R-matrix is given by

$$R_{12}(u) = u + \eta P_{12}, \quad P|i,j\rangle = |j,i\rangle, \quad i,j = 1,\ldots,n, \tag{47}$$

and the associated the most general K-matrices are given by

$$K^-(u) = \xi + uM, \quad M^2 = \mathrm{id}, \tag{48}$$

$$K^+(u) = \bar{\xi} - (u + \frac{n}{2}\eta)\bar{M}, \quad \bar{M}^2 = \mathrm{id}, . \tag{49}$$

The R-matrix satisfies QYBE and the K-matrices satisfy REs. The transfer matrix is given by

$$t(u) = tr_0\left\{K_0^+(u)T(u)K_0^-(u)\hat{T}_0(u)\right\}, \quad [t(u), t(v)] = 0,$$

$$T_0(u) = R_{0N}(u-\theta_N)R_{0\,N-1}(u-\theta_{N-1})\ldots R_{01}(u-\theta_1),$$

$$\hat{T}_0(u) = R_{10}(u+\theta_1)\ldots R_{N-1\,0}(u+\theta_{N-1})\ldots R_{N0}(u+\theta_N).$$

Intrinsic properties of the R-matrix

$$R_{12}(0)=\eta P_{12},\quad R_{12}(\pm\eta)=\pm 2\eta P^{(\pm)}_{12}, \tag{50}$$

$$R_{12}(u)\,R_{21}(-u)=\rho_1(u)\,\mathrm{id},\quad R^{t_1}_{12}(u)\,R^{t_1}_{21}(-u-n\eta)=\rho_2(u)\,\mathrm{id}. \tag{51}$$

and the corresponding properties of the K-matrices:

$$K^{-}(0)=\xi,\quad K^{+}(-\frac{n}{2}\eta)=\bar{\xi}, \tag{52}$$

$$K^{-}(u)K^{-}(-u)\propto \mathrm{id},\quad K^{+}(u)\,K^{+}(-u-n\eta)\propto \mathrm{id}. \tag{53}$$

These intrinsic properties of the R-matrix and K-matrices lead to the operator identities:

$$\begin{aligned} t(\pm\theta_j)t_m(\pm\theta_j-\eta) &= t_{m+1}(\pm\theta_j)\prod_{k=1}^{m}\rho_2^{-1}(\pm 2\theta_j-k\eta)\rho_0(\pm\theta_j), \\ & m=1,\ldots,n-1, \quad j=1,\ldots,N, \\ \rho_0(u) &= \prod_{l=1}^{N}(u-\theta_l-\eta)(u+\theta_l-\eta)\prod_{k=2}^{m}(2u-k\eta)(-2u-k\eta+(n-2)\eta), \\ t_n(u) &= \mathrm{Det}_q(u)\mathrm{id}, \end{aligned} \tag{54}$$

and others $n(n-1)$ relations among $\{t_m(u)\}$. The above relations completely determine the eigenvalues of all fused transfer matrices.

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The R-matrix reads

$$R_{12}(u)=\left(\begin{array}{ccc|ccc|ccc}
c(u) & & & & & & & & \\
 & b(u) & & e(u) & & & & & \\
 & & d(u) & & g(u) & & f(u) & & \\
\hline
 & \bar{e}(u) & & b(u) & & & & & \\
 & & \bar{g}(u) & & a(u) & & g(u) & & \\
 & & & & & b(u) & & e(u) & \\
\hline
 & & \bar{f}(u) & & \bar{g}(u) & & d(u) & & \\
 & & & & & \bar{e}(u) & & b(u) & \\
 & & & & & & & & c(u)
\end{array}\right).$$

It is the first simplest model beyond A-type.

The matrix elements are

$$a(u) = \sinh(u-3\eta) - \sinh 5\eta + \sinh 3\eta + \sinh \eta, \quad b(u) = \sinh(u-3\eta) + \sin 3\eta,$$

$$c(u) = \sinh(u-5\eta) + \sinh \eta, \quad d(u) = \sinh(u-\eta) + \sinh \eta,$$

$$e(u) = -2e^{-\frac{u}{2}} \sinh 2\eta \cosh(\frac{u}{2} - 3\eta), \quad \bar{e}(u) = -2e^{\frac{u}{2}} \sinh 2\eta \cosh(\frac{u}{2} - 3\eta),$$

$$f(u) = -2e^{-u+2\eta} \sinh \eta \sinh 2\eta - e^{-\eta} \sinh 4\eta,$$

$$\bar{f}(u) = 2e^{u-2\eta} \sinh \eta \sinh 2\eta - e^{\eta} \sinh 4\eta,$$

$$g(u) = 2e^{-\frac{u}{2}+2\eta} \sinh \frac{u}{2} \sinh 2\eta, \quad \bar{g}(u) = -2e^{\frac{u}{2}-2\eta} \sinh \frac{u}{2} \sinh 2\eta.$$

The associated non-diagonal K-matrices $K^{-}(u)$ is

$$K^{-}(u) = \begin{pmatrix} 1+2e^{-u-\epsilon}\sinh\eta & 0 & 2e^{-\epsilon+\sigma}\sinh u \\ 0 & 1-2e^{-\epsilon}\sinh(u-\eta) & 0 \\ 2e^{-\epsilon-\sigma}\sinh u & 0 & 1+2e^{u-\epsilon}\sinh\eta \end{pmatrix},$$

$$K^{+}(u) = \mathcal{M}K^{-}(-u+6\eta+i\pi)\big|_{(\epsilon,\sigma)\to(\epsilon',\sigma')},$$

$$\mathcal{M} = \mathrm{Diag}(e^{2\eta},1,e^{-2\eta}).$$

There four boundary parameters ϵ, σ, ϵ', σ'.

Intrinsic properties of the R-matrix

$$R_{12}(0) \propto \eta P_{12}, \quad R_{12}(u)\, R_{21}(-u) = \rho_1(u), \tag{55}$$

$$R_{12}^{t_1}(u)\, \mathcal{M}_1\, R_{21}^{t_1}(-u+12\eta)\, \mathcal{M}_1^{-1} = \rho_2(u)\ \times \mathrm{id}, \tag{56}$$

$$R_{12}(6\eta + i\pi) = P_{12}^{(1)} \times S_{12}^{(1)}, \quad R_{12}(4\eta) = P_{12}^{(3)} \times S_{12}^{(3)}. \tag{57}$$

and the corresponding properties of the K-matrices:

$$K^-(0) = \xi, \quad K^+(-\frac{n}{2}\eta) = \bar{\xi}, \tag{58}$$

$$K^-(u)K^-(-u) \propto \mathrm{id}, \quad K^+(u)\, K^+(-u+6\eta+i\pi) \propto \mathrm{id}. \tag{59}$$

$P_{12}^{(1)}$ and $P_{12}^{(3)}$ are projectors with rank 1 and 3 respectively.

These properties lead to the following operator identities which complete characterize the spectrum of the transfer matrix

$$t(\theta_j)t(\theta_j+6\eta+i\pi) = \frac{\delta_1(u)\ \times \mathrm{id}}{\rho_1(2u)}\Big|_{u=\theta_j}, \quad j=1,\ldots,N, \tag{60}$$

$$t(\theta_j)t(\theta_j+4\eta) = \frac{\delta_2(u)\ \times t(u+2\eta+i\pi)}{\rho_2(-2u+8\eta)}\Big|_{u=\theta_j}, \quad j=1,\ldots,N, \tag{61}$$

$$t(u) = t(-u+6\eta+i\pi), \quad t(u)=t(u+2i\pi), \tag{62}$$

and the values of the transfer matrix at 0, $i\pi$, ∞.

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Conclusion and comments

So far, many typical $U(1)$-symmetry-broken models have been solved by the method:

- The spin torus.
- The XYZ closed spin chain.
- The spin-$\frac{1}{2}$ Heisenberg chain with arbitrary boundary fields and its higher spin generalization.
- The τ_2-model which related to the relativistic Toda chain.
- The open spin chains with general boundary condition associated with A-type algebras.
- The Hubbard model with unparallel boundary fields.
- The t-J model with unparallel boundary fields.
- The Izergin-Korepin model with non-diagonal boundary terms.

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Integrability $\Leftrightarrow$ Solvability

Thank for your attentions