Scattering equations

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Prelude

• Scattering amplitudes are basic observables.

• They are important for LHC.

• The tree level amplitudes are the same for all gauge theories.

• They have hidden structures and simplicity.

• They are complicated to be evaluated with traditional methods, but the answer can be surprisingly simple.

• They are related to other areas of physics and mathematics, like integrability, graph theory, combinatorics, twistors, Wilson loops etc.

• Several methods have been developed the last years like recursions relations, unitarity methods, etc.

Introduction

- Perhaps one can say that in 2003 Witten revolutionized the study of scattering amplitudes by connecting string theory, twistor space and $\mathcal{N} = 4$ super Yang-Mills into a twistor string theory.

- In Witten's theory tree-level field theory amplitudes are given as integrals that localize on generic spheres.

- Most of the developments in the past decade have been focused on particular theories such as $\mathcal{N} = 4$ super Yang-Mills and $\mathcal{N} = 8$ supergravity.

- The natural question is: what is the space of all field theories whose S-matrix can be expressed as an integral over the moduli space of punctured sphere.

- There seems to be that the key ingredient for massless particles is the so-called scattering equations.

Outline

We will

- **define the scattering equations**
- **discuss the properties of the scattering equations**
- **evaluate explicit n-point amplitudes for special kimematics**
- **evaluate some amplitudes for general kinematics**

Scattering equations

The scattering equations define a map from the space of kinematic invariants $k_a \cdot k_b$ to the moduli space of punctured spheres. The explicit form is

$$\boxed{\sum_{b \neq a} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0,}$$

where k_c is the momentum and σ_c the position of the c^{th} puncture.

The scattering equations have appeared before

- in Fairlie and Roberts (1972) in the early days of dual models when they were seeking a variation of the Veneziano model that was free of tachyons
- in Gross and Mende (1988) at the high energy behavior of string theory
- in different contexts in works of Witten, Cachazo, etc.

Properties of the scattering equations

Some comments on the scattering equations

$$\boxed{f_a \equiv \sum_{b\neq a} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0.}$$

- They are n of them.
- They are invariant under $\sigma_a \to \frac{a\sigma_a + b}{c\sigma_a + d},\ ad - bc = 1.$
- Only $n-3$ of the σ's are independent. We can fix three of them to arbitrary values, for example $\sigma_1 = 1,\ \sigma_2 = -1,\ \sigma_3 = \infty.$ They satisfy $\sum_a f_a = \sum_a \sigma_a f_a = \sum_a \sigma_a^2 f_a = 0.$
- They have $(n-3)!$ solutions.
- σ's can be complex.
- They admit a polynomial for in $n-3$ variables.
- They are deceivingly simple. Apart from special kinematics, no solutions are known beyong $n = 5$.

Use of the scattering equations

The modern use of the scattering equations (Cachazo-He-Yuan) is to allow us to express the tree level in arbitrary spacetime dimensions of a vast number of massless theories in a nice way. The general form of all these amplitudes is

$$\boxed{\text{Amplitude} = \int d^n\sigma \; I_n(\epsilon, p, \sigma) \prod_a \delta(f_a),}$$

where ϵ is the helicity and p the momentum of the external particles. I_n depends on the theory under consideration and it contains information about the kinematics and helicities.

The delta function completely localizes the integral and we can write

$$\boxed{\text{Amplitude} = \sum_{\text{solutions}} I_n / \left| \frac{\partial f_a}{\partial \sigma_b} \right| .}$$

Examples

We give some examples. For Yang-Mills we have

$$I_{\text{Yang-Mills}} = \frac{\mathrm{Pf}\Psi}{\sigma_{12}\sigma_{23}\cdots\sigma_{n1}},$$

where $\sigma_{ij} = \sigma_i - \sigma_j$ and Ψ is a $2n \times 2n$ antisymmetric matrix given by

$$\Psi = \begin{pmatrix} A & -C^{\mathrm{T}} \\ C & B \end{pmatrix},$$

and

$$A_{ab} = \frac{k_a \cdot k_b}{\sigma_a - \sigma_b}, \quad B_{ab} = \frac{\epsilon_a \cdot \epsilon_b}{\sigma_a - \sigma_b}, \quad C_{ab} = \frac{\epsilon_a \cdot k_b}{\sigma_a - \sigma_b}.$$

Some more examples

$$I_{\phi^3} = \frac{1}{\sigma_{12}\sigma_{23}\cdots\sigma_{n1}},$$
$$I_{\text{Yang-Mills}} = \frac{\mathrm{Pf}\Psi}{\sigma_{12}\sigma_{23}\cdots\sigma_{n1}},$$
$$I_{\text{gravity}} = \det\Psi.$$

We can see that

$$\boxed{\text{gravity} \times \text{scalar} = (\text{Yang Mills})^2.}$$

- More theories are known, like Einstein-YM, YM-scalar, DBI, NLSM, etc.
- All the above theories are massless and do not contain fermions.
- For Yang-Mills the formula was proven by Dolan-Goddard, by showing that the equations satisfy known recursion relations.

An identity for the roots of polynomials

Since the general problem is hard to solve we will study amplitudes at special kinematics.

Consider a polynomial y with n roots denoted by x_j

$$y = k \prod_{j=1}^{n} (x - x_j)$$

that satisfies the second order differential equation

$$ay'' + by' + cy = 0.$$

We take the logarithm and then the derivative of the polynomial to get

$$\frac{y'}{y} = \sum_{j=1}^{n} \frac{1}{x - x_j}.$$

An identity for the roots of polynomials

We isolate the i^{th} root

$$\frac{y'}{y} = \sum_{j\neq i}^{n} \frac{1}{x - x_j} + \frac{1}{x - x_i},$$

or equivalently

$$\sum_{j\neq i}^{n} \frac{1}{x - x_j} = \frac{(x - x_i)y' - y}{(x - x_i)y}.$$

We take the limit where x approaches x_i

$$\sum_{j\neq i}^{n} \frac{1}{x_i - x_j} = \frac{y''}{2y'} = -\frac{b}{2a},$$

where we have used the second order differential equation that the polynomial satisfies.

Determination of the special kinematics

We compare the scattering equations with the equation that the roots of polynomials satisfy

$$\sum_{b\neq a}\frac{k_a\cdot k_b}{\sigma_a-\sigma_b}=0 \qquad \text{vs} \qquad \sum_{j\neq i}^{n}\frac{1}{x_i-x_j}=-\frac{b}{2a}$$

and we claim they are the same if one chooses some special kinematics and polynomials. We choose

- Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and
- $k_1\cdot k_a=(1+\beta)/2,\ k_2\cdot k_a=(1+\alpha)/2,\ k_a\cdot k_b=1,\ a,b\geq 4.$

Evaluation of the amplitude

For the above special kinematics we can evaluate the amplitudes for various theories. For example for Yang-Mills we find that

$$\boxed{A_n = \frac{P^{(\frac{\alpha-1}{2},\frac{\beta-1}{2})}_{\frac{n}{2}-1}(1)}{P^{(\frac{\alpha}{2},\frac{\beta}{2})}_{\frac{n}{2}-2}(-1)} \times \text{helicities.}}$$

- The above amplitude can be rewritten in many different forms including Gamma functions and factorials.
- The Jacobi polynomials satisfy recurrence relations that look like $P_n = P_{n-1} + P_{n-2}$, that hint to recurrence relations for the scattering amplitudes.
- One can use different polynomials to study different kinematics.
- For gravity the result also exists.
- One idea is to associate the amplitude to an integrable system. Our construction has elements of integrability.

Proof of the relations for the special kinematics

One can evaluate any scattering amplitude for our special kinematics using the following arguments.

• For our special kinematics the scattering equations can be written as $n-3$ equations that all look like

$$a_1 s_1 + a_2 s_2 + a_3 s_3 + \ldots = b,$$

where s_i are the symmetric polynomials and a_i and b constants. For example

$$\begin{aligned} s_1 &= \sigma_1 + \sigma_2 + \sigma_3 + \ldots, \\ s_2 &= \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3 + \ldots, \\ s_3 &= \sigma_1\sigma_2\sigma_3 + \ldots, \\ &\vdots \end{aligned}$$

The system is linear and we can easily solve for s_i.

Proof of the relations for the special kinematics

• Due to symmetry if

$$(\ldots, \sigma_i, \ldots, \sigma_j, \ldots)$$

is a solution of the scattering equations, so does

$$(\ldots, \sigma_j, \ldots, \sigma_i, \ldots).$$

• This means that if we know one of the solutions of the scattering equation, for example

$$(\sigma_1, \sigma_2, \sigma_3, \ldots)$$

then we can get all of them by simply permuting the above solution.

• Then any amplitude can be evaluated explicitly as

$$\begin{aligned}\text{Amplitude} &= \sum_{\text{solutions}} f(\sigma_1, \sigma_2, \sigma_3, \ldots) \\ &= \sum_{\sigma_i \ \text{perms}} f(\sigma_1, \sigma_2, \sigma_3, \ldots) \\ &= F(s_1, s_2, s_3, \ldots).\end{aligned}$$

General kinematics - a toy model

For general kinematics we cannot solve the scattering equations, but fortunately one does not need to do so in order to evaluate a scattering amplitude. In order to see why this is the case, let us consider the toy scattering equation

$$x^2 - ax + b = 0.$$

We know that the sum of roots of the above polynomial scattering equation is a, whereas the product of roots is b, because

$$(x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1 r_2 = 0.$$

Then we can just evaluate any toy model scattering amplitude as a function of a and b without the need of knowing the explicit solutions of the toy scattering equation. We illustrate this idea with a few simple examples.

General kinematics - a toy model

We consider a few toy amplitudes that satisfy the toy model scattering equation

$$x^2 - ax + b = 0.$$

- First example

$$A = \sum_{\text{roots}} x^2 = r_1^2 + r_2^2 = (r_1 + r_2)^2 - 2r_1 r_2 = a^2 - 2b.$$

- Second example

$$A = \sum_{\text{roots}} \frac{1}{x^2} = \frac{1}{r_1^2} + \frac{1}{r_2^2} = \frac{r_1^2 + r_2^2}{r_1^2 r_2^2} = \frac{a^2 - 2b}{b^2}.$$

This holds in general

$$A = \sum_{\text{roots}} f(x) = F(a, b).$$

All five point tree amplitudes

Using the above ideas we would like to evaluate the first non-trivial case, namely all the five point amplitudes at tree level. The claim is that they are linear combinations of the following fundamental quantity

$$\boxed{P = \sum_{\text{roots}} \frac{\text{cross ratios of sigmas}}{\text{Jacobian of scattering equations}}.}$$

The reason this claim is true is that P is the most general quantity allowed by symmetries. For five points we have five cross ratios. More explicitly we have

$$P = \sum_{\text{roots}} \frac{1}{\text{Jacobian}} \prod_{i=1}^{5} \left(\frac{\sigma_{i,i+1}\sigma_{i+2,i+3}}{\sigma_{i,i+2}\sigma_{i+1,i+3}} \right)^{\alpha_i}$$

and

$$A = \sum \text{coefficient } P,$$

where the sums runs over multiplicities of the cross ratios.

All five point tree amplitudes

The goal is to give an explicit expression of the fundamental quantity P. The answer is in general complicated but it seems there is a way to put an order. We have found that one can express P through the introduction of a generating function G

$$\boxed{P = \left(\prod_{i=1}^{5} \frac{1}{\alpha_i!}\frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}\right) G(x_i)\Big|_{x_i=0}\,,}$$

where

$$G(x_i) = \frac{\sum_{i<j<k<l<m}^{5}(d_0 + d_i x_i + d_{ij} x_i x_j + \ldots + d_{ijklm} x_i x_j x_k x_l x_m)}{\prod_{i=1}^{5}(1 + b_i x_i + c_i x_i^2)}.$$

The coefficients appearing above are simple functions of the momentum, for example $b_0 = \sum_{i=1}^{5} \frac{1}{(k_i \cdot k_{i+1})(k_{i+2} \cdot k_{i+3})}$.

General comments

• One can apply the same ideas and techniques to the n-point case. The form of the generating function in the general case is easy to find. The difficult part of the calculation is the express the coefficients of the generating function in terms of kinematic invariants.

• In the meanwhile at least two more independent methods have been devoloped that touch on the same problem. They use graph theory, the global residue theorem and other ideas.

Summary

- We have used the scattering equations to explicitly evaluate gluon and graviton n-point amplitudes for special kinematics. The answer is given in terms of Jacobi polynomials.
- We have explained how to evaluate amplitudes for the special kinematics for other theories.
- We have presented an algorithm that evaluates all amplitudes for general kinematics from the coefficients of the scattering equations.
- We have explicitly evaluated all n-point amplitudes for general kinematics through the introduction of a generating function.

To be understood

- Which theories admit an S-matrix representation in terms of scattering equations?
- How can we explicitly evaluate all amplitudes?
- Is there a simple formula for them?
- What happens at loop level?
- What happens to theories with fermions?
- ...