Near Horizon Structure of Extremal Vanishing Horizon Black Holes

(3 Theorems for near horizon geometries)

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- Sadeghian, Sheikh-Jabbari, Vahidinia and H. Y , *to be appeared*.
- Sadeghian, Sheikh-Jabbari, Vahidinia and H. Y *ArXiv* : 1504.03607
- Sadeghian, Sheikh-Jabbari, H.Y, *JHEP* 10, 081 (2014)

1. Motivation
   - Black Holes & Quantum Gravity
   - Gauge/gravityduality
   - Black Hole Classification
2. General EVH and NHEVH ansatz
   - Near Horizon Geometry
   - General implications of Einstein equations
3. 3 Theorems on near horizon geometries
   - Theorem 1
   - Theorem 2
   - Theorem 3
4. Conclusion

Black Hole as the key to Quantum Gravity (Q.G)

- Research on B.H is relevant and interesting
  - *Black hole solutions (BH) to Einstein's equations have been known since the advent of general relativity.*
  - *The most obvious reason such solutions are of physical interest is the expectation that they arise as the end state of catastrophic gravitational collapse of some suitably localised matter distribution.*
  - *A less obvious reason such solutions are important is that they have played a key role in guiding studies of quantum gravity.*
- There are various ways B.Hs and their near-horizon geometries have appeared in modern studies of quantum gravity.

AdS/CFT

- A significant breakthrough in the study of quantum gravity is the Anti de Sitter/Conformal Field Theory (AdS/CFT) duality.
- In principle, AdS/CFT asserts a fully non-perturbative equivalence of quantum gravity in asymptotically AdS spacetimes with a conformally invariant quantum field theory in one lower spatial dimension.
- This is an explicit realisation of a *holographic principle* underlying quantum gravity.

AdS_3/CFT_2

- One of the best known examples is duality between BHs in AdS_3 space (BTZ BHs) and 2-dim CFT.
- A recently-developed approach describes more general BHs; a CFT_2 description has been proposed for a certain class of near-extremal BH, which possess a local near-horizon AdS_3 (and AdS_n with $n > 3$) factor but a vanishing horizon area in the extremal limit.
- EVH/CFT

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Black Hole Classification

The classification of higher-dimensional stationary B.H solutions is a major open problem in G.R. (and Diff.Ge.)

- In 4-d the B.H uniqueness theorem provides an answer to the classification problem for asymptotically flat B.H of E-M theory.
- Generalisation to higher dimensions: positive Yamabe condition, topological censorship theorem, symmetries arguments.
- The general topology and symmetry constraints discussed above become increasingly weak as one increases the number of dimensions and there are evidences that B.H uniqueness will be violated much more severely as one increases the dimensions.
- In the absence of new ideas, it appears that the general classification problem for even asymptotically-flat B.Hs is hopelessly out of reach.

Black Hole Classification

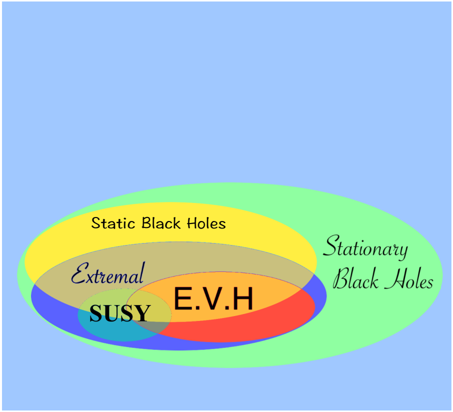
The classification of higher-dimensional stationary B.H solutions is a major open problem in G.R. (and Diff.Ge.)

- Asymptotically AdS BHs: A further complication in AdS comes from the choice of asymptotic boundary conditions.
- It is clear that supersymmetry provides a technically-simplifying assumption to classifying spacetimes.
- It turns out that extremality can also be used as a simplifying the problem.
- Classification of near-horizon geometries

Black Hole Classification

- Exploring the main issues that appear in the general BH classification problem, such as the horizon topology, spacetime symmetry and the number of solutions.
- The main drawback of this approach is that the existence of a near-horizon geometry solution does not guarantee the existence of a corresponding BH solution.
- One can use this method to rule out possible BH horizon topologies, for if one can classify near-horizon geometries completely and a certain horizon topology does not appear, this implies there can be no extremal BH with that horizon topology either.
- A notable example of this method has been a proof of the non-existence of supersymmetric AdS black rings in $D = 5$ minimal gauged supergravity

EVH Black Holes



EVH black holes

- Vanishing T, A_h but with T/A_h = fixed.
- EVH B.H are not small black holes (generically points on the horizon are at a finite distance from singularity),
- Extremal B.H are identified by an n-1 dimensional hypersurface, in n-dimensional B.H parameter space,
- Vanishing horizon area condition may impose more than one relation among parameters,
- EVH B.H are a subclass of extremal ones and are defined by n-p dim, p$\geq$ 2.

Examples of EVH Black Holes

- n=2, p=2: Massless BTZ
  Jan de Boer, M.M. Sheikh-Jabbari, Joan Simon,
  Class.Quant.Grav.28:175012,2011
- n=3, p=2: 5d Kerr B.H with one vanishing J
  H.Y, *unpublished note*
- n=4, p=2: 4d KK B. H
  M.M. Sheikh-Jabbari, H.Y **JHEP 1110 (2011) 013**
  2-charge static AdS_5 B. H
  J. de Boer, M. Johnstone, M.M. Sheikh-Jabbari, Joan Simon
  Phys.Rev. D85 (2012) 084039
  3-charge static B.H in 5d Heterotic String
  H.Y, **PEur.Phys.J. C72 (2012) 2256**

Examples of EVH Black Holes

- n=4, p=3: Rotating-charged AdS_5 B.H with 2 equal charges
  M. Johnstone, M.M. Sheikh-Jabbari, Joan Simon and H.Y
  JHEP 1304 (2013) 0459
- n=5, p=2: 3-charg AdS_4 B.H
  R. Fareghbal, C.N. Gowdigere, A.E. Mosaffa, M.M. Sheikh-Jabbari
  Phys.Rev. D81 (2010) 046005
  4-charge static B.H in 4d Heterotic String
  H.Y, Eur.Phys.J. C72 (2012) 2256
- etc.

- EVH black holes can be:
  - supersymmetric or non-BPS,
  - stationary and static,
  - asymptotically flat, dS or AdS.

1. AdS$_3$ throat in the near horizon geometry.
2. Near horizon limit is a decoupling limit.
   NOTE: Decoupling condition $\Rightarrow$ to unitarity condition of the 2d CFT.
   Although there an AdS$_2$ throat, the near horizon limit of usual extremal B.H is not necessarily a decoupling limit.
3. Entropy and Temperature of the original near-EVH black hole and the BTZ in near horizon geometry are equal.

EVH/CFT Correspondence

Gravity theory on the near horizon limit of EVH black holes is governed by a 2d CFT.

`M. M. Sheikh-Jabbari and H.Y, JHEP 1110 (2011)`

In the vicinity of any degenerate Killing horizon the metric in the Gaussian null coordinates can be written as:

$$ds^2 = 2drdv + 2r\mathrm{f}_i(r,y)dvdy^i - r\mathrm{F}(r,y)dv^2 + \mathrm{h}_{ij}(r,y)dy^i dy^j .$$

- $N = \partial_v$ is a Killing vector field which becomes null at the outer horizon located at $r = 0$; ∂_v is the vector field which creates the outer Killing horizon. The Gaussian null coordinates covers the region outside the Killing horizon and is constructed such that ∂_r is a vector field which is null everywhere.
- Entropy and Hawking temperature are given by

$$T_H = \frac{1}{4\pi}\mathrm{F}(r=0), \quad A_h = \int_{r=0} \sqrt{\det \mathrm{h}}\, d^{d-2}y \,.$$

- For an EVH black hole then $T_H \sim A_h \sim \epsilon$ where ϵ is a small parameter measuring how close to extremality we are.

- Assume vanishing of A_h is due to a vanishing one-cycle: at $r=0$ and (parameterize this direction by ϕ and take ∂_ϕ to be a Killing direction). It is then more convenient to decompose y^i into (x^a,ϕ). The leading ϵ expansion of the metric functions hence take the form

$$\mathrm{F}(r,y)=\epsilon F^{(1)}+rF(x) \quad \text{and}$$
$$\mathrm{h}_{ij}dy^i dy^j=\mathrm{G}(r,x)d\phi^2+2\mathrm{g}_a(r,x)d\phi dx^a+\hat{\gamma}_{ab}(r,x)dx^a dx^b$$
$$\mathrm{g}_a=\epsilon g_a^{(1)}(x)+rg_a(x),\ \mathrm{G}=\epsilon^2 G^{(2)}(x)+\epsilon rG^{(1)}+r^2G(x)$$

Along with $\epsilon \to 0$, we take the near horizon limit:

$$r \to \lambda r, \quad v \to \frac{v}{\lambda}, \quad \phi \to \frac{\phi}{\lambda}, \qquad \lambda \to 0,$$

to obtain

$$ds^2 = -r\left(\frac{\epsilon}{\lambda}F^{(1)} + rF\right)dv^2 + 2r\left(\frac{\epsilon}{\lambda}H^{(1)} + rH\right)d\phi dv + \left(\frac{\epsilon^2}{\lambda^2}G^{(2)} + \frac{\epsilon}{\lambda}rG^{(1)} + r^2G\right)d\phi^2 + 2drdv + 2rf_a dx^a dv + 2\left(\frac{\epsilon}{\lambda}g_a^{(1)} + rg_a\right)dx^a d\phi + \gamma_{ab}dx^a dx^b + \mathcal{O}(\lambda,\epsilon)\,.$$

- If we take this limit such that $\epsilon \ll \lambda$ we are dealing with the near-horizon EVH geometry.
- while taking the limit $\epsilon \sim \lambda$ correspond to near-EVH near-horizon limit.
- The $\epsilon \gg \lambda$ case (while $\epsilon \to 0$) corresponds to far from EVH cases and we do not discuss it here.

Taking near horizon limit of EVH black hole solutions gives

$$ds^2 = r^2\left[-Fdv^2 + Gd\phi^2 + 2Hd\phi dv\right] + 2drdv$$

$$+2r\left[f_a dx^a dv + g_a dx^a d\phi\right] + \gamma_{ab}dx^a dx^b$$

- To restrict further the form of this metric we use smoothness properties and Einstein equations which in d dimensions take the form

$$R_{\mu\nu} = T_{\mu\nu} + \frac{2\Lambda - T}{d-2} g_{\mu\nu}$$

where $R_{\mu\nu}$, $T_{\mu\nu}$ respectively denote Ricci curvature and energy-momentum tensor, T is the trace of energy-momentum tensor and Λ is the cosmological constant.

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Theorem 1

Near horizon of EVH black hole solutions in Einstein gravity coupled with matter fields which have finite and analytic energy momentum tensor at the horizon and $T_{\phi a} = T_{va} = 0$, have a three dimensional locally maximally symmetric part.

Proof

Recalling that in the Gaussian null coordinates g_{rr}, g_{ra} components of the NHEVH metric ansatz are zero, smoothness and analyticity of the energy-momentum tensor at the horizon at $r = 0$, which is a generic feature of black hole solutions, implies

$$R_{rr} = 0, \qquad R_{ra} = 0.$$

These imply that $g_a = 0$ and $f_a = \partial_a G/G$. Next, if we also assume that T_{va} and $T_{\phi a}$ vanish for the near horizon EVH geometry, Einstein equations restrict the form of metric to:

$$ds^2 = e^{-2K}\left[A_0\rho^2 dv^2 + 2dvd\rho + \rho^2 d\phi^2\right] + \gamma_{ab}dx^a dx^b$$

- $A_0 > 0$: locally dS_3
- $A_0 = 0$: locally $M^{(1,2)}$
- $A_0 < 0$: locally AdS_3

- A general theory of Einstein theory in arbitrary dimensions coupled to scalar and gauge fields, including all gauged and un-gauged supergravity satisfy conditions ofTheorem 1.

Theorem 2

Theorem 2

In theories of gravity coupled with matter fields satisfying strong energy condition, and with non-positive cosmological constant Λ, the 3d part of near horizon of an EVH black is AdS_3 for $\Lambda < 0$ and for $\Lambda = 0$ either it is an AdS_3 or the geometry is a direct product of a locally 3d flat space and a $d-3$ dimensional part.

Proof

Proof:

The strong energy condition stipulates that:
$(T_{\mu\nu} - \frac{1}{d-2} T g_{\mu\nu}) t^\mu t^\nu \geq 0$ (for every future-oriented timelike vector field t^μ)
Eliminating $T_{\mu\nu}$ from the Einstein equations for metric we arrive at

$$\nabla^2 K - 3(\nabla K)^2 - \frac{2\Lambda}{d-2} + 2A_0 e^{2K} \leq 0$$

where ∇^2 denotes the Laplacian computed with metric γ_{ab} and $(\nabla K)^2 = \gamma^{ab}\partial_a K \partial_b K$. Multiplying it by $e^{-\alpha K}$ with $\alpha \geq 3$ and integrating it on $d-3$ dimensional part, when γ has a finite volume we get

$$\int_{\gamma_{d-3}} d^{d-3}x \sqrt{\det\gamma}\, e^{-\alpha K} \left[\frac{\alpha-3}{2}(\nabla K)^2 - \frac{\Lambda}{d-2} + e^{2K} A_0 \right] \leq 0.$$

Therefore, if $\partial_a K \neq 0$ then $A_0 < 0$ for any $\Lambda \leq 0$ and the near-horizon EVH geometry contains an AdS_3 factor. The flat 3d case, $A_0 = 0$, is only possible when $K = const$. and $\Lambda = 0$, where the warp factor e^{-K} becomes is a constant. For $\Lambda > 0$ cases, the above analysis does not yield a restriction on the sign of A_0. □

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Theorem 3

In theories with non-positive cosmological constant, the 3d part of near horizon of a near-EVH black hole is either a BTZ black hole or a rotating massive particle on the flat spacetime.

Proof

Proof:

The near-EVH near-horizon: $\epsilon \sim \lambda$

Let us define parameter $\alpha = \epsilon/\lambda$ measures *"out-of-EVH-ness"* and $\alpha = 0$ corresponds to the EVH point.

Since these equations should be valid for arbitrary α in the given range, these equations may be expanded in powers of α.

One then has the zeroth order EVH ($\alpha = 0$) results to obtain:

$$ds^2 = e^{-2K}\left[-\rho(\rho\tilde{F}+\alpha F^{(1)})dv^2+2\rho(\tilde{H}\rho+\alpha H^{(1)}(x))dvd\phi+2dvd\rho+[(\rho+\alpha R(x))^2+\alpha^2 J(x)]d\phi^2\right]$$

$$+2\alpha g_a^{(1)}(x)dx^a d\phi + \gamma_{ab}(x)dx^a dx^b,$$

where $\tilde{F}$, $\tilde{H}$ are constants by virtue of zeroth α order equations, while $F^{(1)}$ is a constant because it is related to the surface gravity of the near-EVH black hole.

Noting that $|\partial_\phi|^2 > 0$ and that *detg* should not change sign, we learn

$$J \geq 0, \quad (\rho+\alpha R)^2 + \alpha^2 J > \alpha^2 \gamma^{ab} g_a^{(1)} g_b^{(1)}$$

Theorem 3

smoothness and analyticity of $T_{\mu\nu}$ at $r = 0$ imply: $T_{\rho\rho} = T_{\rho a} = 0 \Longrightarrow \mathrm{R}_{\rho\rho} = R_{\rho a} = 0$. which in turn yields

$$J = \gamma^{ab} g_a^{(1)} g_b^{(1)}$$

With the above, the metric may be written as

$$ds^2 = e^{-2K}\Big[-\rho(\rho\tilde{F} + \alpha F^{(1)})dv^2 + 2\rho(\tilde{H}\rho + \alpha H^{(1)})dv d\phi + 2dv d\rho + (\rho + \alpha R)^2 d\phi^2 \Big]$$

$$+\gamma_{ab}(dx^a + \alpha\hat{g}^a d\phi)(dx^b + \alpha\hat{g}^b d\phi) \ ,$$

If we assume $\hat{g}^a = 0$, analysis of equations of motion and in particular with the $T_{va} = 0$, $T_{\phi a} = 0$ imply R, $H^{(1)}$ are constants.
This assumption is equivalent to ∂_ϕ be a hypersurface orthogonal Killing vector on the horizon of the original EVH black hole; i.e. at codimension two constant v and $r = 0$ surfaces, ∂_ϕ is transverse to the constant ϕ surfaces.

From components of the Einstein equations along the 3d part:

$$H^{(1)} = 2\tilde{H}R\,, \qquad F^{(1)} = 2\tilde{F}R$$

If the matter fields satisfy strong energy condition, we deal with two options:

- $A_0 = -(\tilde{F} + \tilde{H}^2) < 0$

$$ds^2 = e^{-2K}\Big[-\tilde{F}\rho(\rho+2\alpha R)dv^2 + 2\tilde{H}\rho(\rho+2\alpha R)dvd\phi + (\rho+\alpha R)^2 d\phi^2 + 2dvd\rho\Big] + \gamma_{ab}dx^a dx^b$$

Its 3d part denotes a BTZ geometry, with inner and outer horizon radii $r_\pm$ and AdS_3 radius ℓ

$$\ell^2 = -\frac{1}{A_0}\,, \qquad r_+ = \alpha R\,, \qquad \tilde{H} = \frac{r_-}{\ell r_+}$$

We note that if the ϕ direction in the original EVH black hole (before taking the near horizon limit) was ranging over $[0, 2\pi]$, after taking the near horizon limit the ϕ direction will be ranging over $[0, 2\pi\lambda]$. This geometry is hence called "pinching BTZ".

Theorem 3

- For $A_0 = 0$

after the shift $\rho \to \rho - \alpha R$ and $\phi \to \phi - \tilde{H}v$, and then rescaling v, ϕ and ρ, metric takes the form

$$ds^2 = e^{-2K}\left[dv^2 + \frac{2}{\tilde{H}}dvd\phi + \rho^2 d\phi^2 + 2dvd\rho\right] + \gamma_{ab}dx^a dx^b$$

where the ϕ coordinate is ranging over $[0, 2\pi\alpha\tilde{H}R\lambda]$. The 3d part of metric is locally flat and represents a particle of a given mass and spin proportional to $\tilde{H}$.

Conclusion

- In this work we proved three theorems regarding near horizon limit of (near) Extremal Vanishing Horizon black hole solutions.
- Our results apply to quite generic gravity theories in diverse dimensions.
- Our theorems state that for theories obeying strong energy condition we generically get an AdS_3 factor in our NHEVH geometry.
- While the possibility of 3d flat space is not ruled out by our theorems, we do not know any explicit example which actually realizes this possibility.

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- It would hence be interesting to explore if this possibility can be ruled out through some other properties (e.g. other energy conditions) of the matter fields.
- Our theorems are not uniqueness or classification theorems, neither for the EVH black holes nor for their near horizon limits; they uncover interesting, generic features of NHEVH geometries.
- Our theorems once considered together with analysis of our previous works, provide a classification and uniqueness for four and five dimensional NHEVH solutions to Einstein-Maxwell-Dialton theories.

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- The ϕ direction is a "*pinching*" direction, it is ranging over $\phi \in [0, 2\pi\lambda]$, if the ϕ direction in the original black hole had a $[0, 2\pi]$ range. Then, one should note this fact if based on this pinching AdS_3 one wants to put forward an EVH/CFT correspondence.
- In this respect and recalling our near-EVH theorem (Theorem 3), the EVH case is interesting because, unlike the extremal case, it allows for "excitation" and nontrivial dynamics about the NHEVH geometry.

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