Vertex operators, $\mathbb{C}^3$ curves and the topological vertex

Jianfeng. Wu[1] Jie. Yang[2]

[1] Department of Applied Mathematics and Physics
Beijing University of Technology

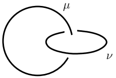
[2] Department of Mathematics Sciences
Capital Normal University

IHEP, August 18, 2014

Motivation

- Duality
  A-model topological string: combinatorical explanation
  B-model topological string: wave function explanation
- Zhou's identity

$$s_\nu(q^{\mu_i - i}) = (-1)^{|\nu|} q^{\kappa_\nu/2} \sum_\eta q^{-|\eta|} \frac{s_{\mu/\eta}(1, q, \cdots)}{s_\mu(1, q, \cdots)} s_{\nu/\eta}(1, q, \cdots) \tag{1}$$

$\Bigg\langle$ 

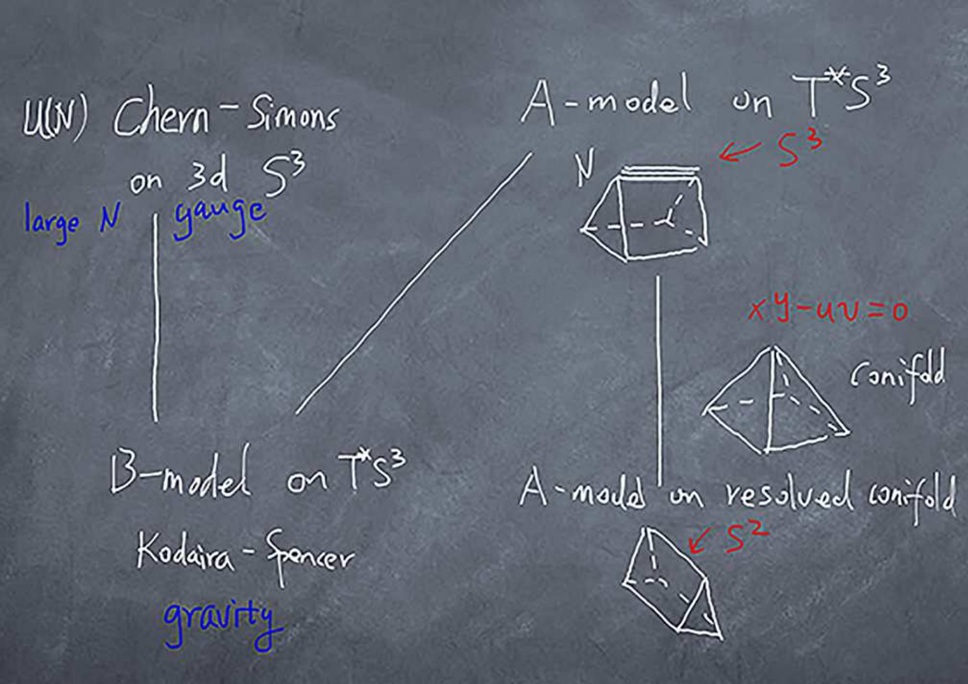
$\Bigg\rangle = W_{\mu\nu} = s_\mu(q^\rho) s_\nu(q^{\mu+\rho})$

where $\rho = -\frac{1}{2}, -\frac{3}{2}, \cdots$.

Outline

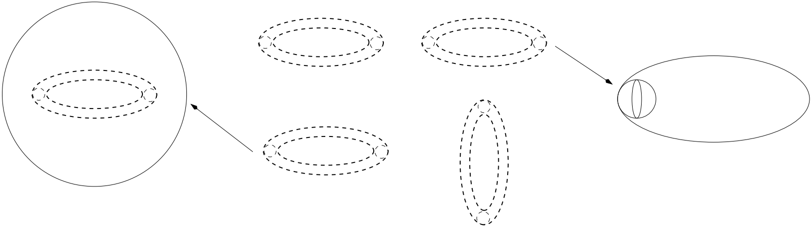
1. Preliminary
2. The technique
3. Future research

Outline


U(N) Chern-Simons
on 3d S³
large N
gauge
A-model on T*S³
N
S³
xy-uv=0
conifold
B-model on T*S³
Kodaira-Spencer
gravity
A-model on resolved conifold
S²

Chern-Simons theory on S^3

- Surgery from S^3 to $S^2 \times S^1$



- Unknot

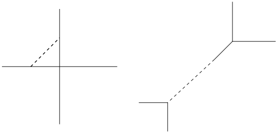
$$Z(S^3, R_i) = \sum_j Z(S^2 \times S^1, R_i, R_j) s_{0j}$$

- Link

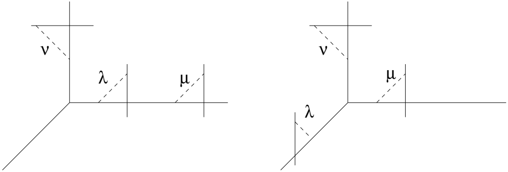
$$Z(S^3, L(R_i; R_j)) = \sum_k Z(S^2 \times S^1, R_k, R_j) s_{ik}$$

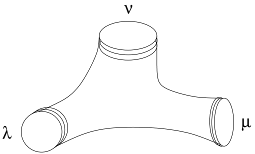
The topological vertex

Conifold transition from T^*S^3 to resolved conifold



With D-branes





Symmetric functions

In this talk the specific symmetric function is called Schur function. It is a generating series for semistandard Young tableaux.

For example

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}.$$

each Young tableaux gives rise to a monomial. Hence $s_{(2,1)}(x_1, x_2, x_3)$ is

$$x_1^2x_2 + x_1x_2^2 + x_1x_2x_3 + x_1x_2x_3 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$$

Definition:

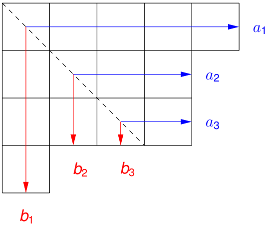
$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta},$$

where

$$a_\alpha = \det x_i^{\alpha_j}, \quad a_\delta = \prod_{i<j}(x_i - x_j)$$

Fermionic excitation and 2D Young diagram

$P(n) = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$ has a fermionic excitation description.



$$\lambda \Leftrightarrow |\lambda\rangle = \prod_{i=1}^{d(\lambda)} \psi_{-(a_i)} \psi^*_{-(b_i)} |0\rangle$$

$$a_i = \lambda_i - i + \frac{1}{2}, \quad b_i = \lambda_i^t - i + \frac{1}{2}$$

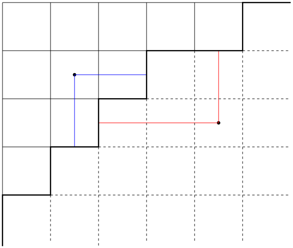
Because of boson-fermion correspondence boson and fermion excitation modes satisfy

$$a_n = \sum_{r \in \mathbb{Z}+1/2} : \psi_{n-r} \psi^*_r : .$$

It is clear how a_n can act on 2D Young diagram.

Outline

Nakajima's formula



$$\prod_{s \notin \lambda} \frac{1}{1 - q^{h(s)}} = \prod_{s \in \lambda} \frac{1}{1 - q^{h(s)}} M(q)$$

where $M(q)$ is the MacMahon function. It is the generating function for 3D Young diagrams namely

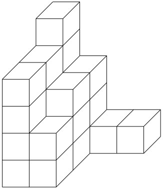
$$M(q) = \prod_{n}^{\infty} \frac{1}{(1 - q^n)^n} = \sum_{\pi \in \text{all } Y_3} q^{|\pi|},$$

where $|\pi|$ counts the number of boxes in 3D Young diagram π.

here

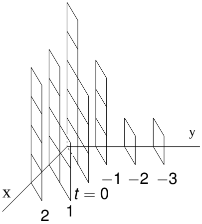
3D Young diagram

- Definition



5	3	1	1
4	3		
4	2		

- The diagonal slicing



Interlacing condition

$$\begin{cases} \lambda(t) \succ \lambda(t+1) & t > 0 \\ \lambda(t) \succ \lambda(t-1) & t < 0 \end{cases}$$

where

$$\lambda \succ \mu \quad \Longleftrightarrow \quad \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \cdots$$

Vertex operators and generating series

- (Half) vertex operators are defined as

$$V_+(z) = \exp\left\{\sum_{n>0} \frac{a_n}{-n} z^{-n}\right\}, \quad V_-(z) = \exp\left\{\sum_{n>0} \frac{a_{-n}}{n} z^{n}\right\},$$

$$V_+^*(z) = \exp\left\{\sum_{n>0} \frac{a_n}{n} z^{-n}\right\}, \quad V_-^*(z) = \exp\left\{\sum_{n>0} \frac{a_{-n}}{-n} z^{n}\right\}.$$

Vertex operators and generating series

- (Half) vertex operators are defined as

$$V_+(z) = \exp\left\{\sum_{n>0} \frac{a_n}{-n} z^{-n}\right\}, \quad V_-(z) = \exp\left\{\sum_{n>0} \frac{a_{-n}}{n} z^{n}\right\},$$

$$V_+^*(z) = \exp\left\{\sum_{n>0} \frac{a_n}{n} z^{-n}\right\}, \quad V_-^*(z) = \exp\left\{\sum_{n>0} \frac{a_{-n}}{-n} z^{n}\right\}.$$

- The generating series are

$$\prod_i V_-(z_i) = \sum_\lambda s_\lambda(z_i) s_\lambda(a_-), \prod_i V_+^*(z_i) = \sum_\lambda s_\lambda(z_i^{-1}) s_\lambda(a_+),$$

Vertex operators and generating series

- (Half) vertex operators are defined as

$$V_+(z) = \exp\left\{\sum_{n>0} \frac{a_n}{-n} z^{-n}\right\}, \quad V_-(z) = \exp\left\{\sum_{n>0} \frac{a_{-n}}{n} z^{n}\right\},$$

$$V_+^*(z) = \exp\left\{\sum_{n>0} \frac{a_n}{n} z^{-n}\right\}, \quad V_-^*(z) = \exp\left\{\sum_{n>0} \frac{a_{-n}}{-n} z^{n}\right\}.$$

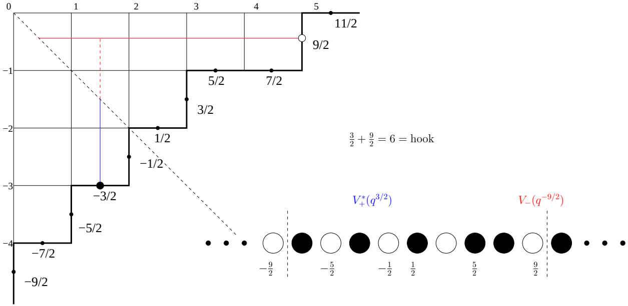
- The generating series are

$$\prod_i V_-(z_i) = \sum_\lambda s_\lambda(z_i) s_\lambda(a_-), \prod_i V_+^*(z_i) = \sum_\lambda s_\lambda(z_i^{-1}) s_\lambda(a_+),$$

- V_- can be treated as the creation operator while V_+^* the annihilation of 2D Young diagrams, i.e.

$$V_-(x)|\mu\rangle = \sum_{\lambda \succ \mu} x^{|\lambda|-|\mu|}|\lambda\rangle, \quad V_+^*(x)|\mu\rangle = \sum_{\lambda \prec \mu} x^{|\lambda|-|\mu|}|\lambda\rangle$$

The A-model: the topological vertex



$$\langle 0| \prod_{\text{profile of } \nu} V_-(q^{-\nu-\rho}) V_+^*(q^{\nu^t+\rho})|0\rangle = \prod_{s\in\nu} \frac{1}{1-q^{h(s)}}$$

Switching $V_- \leftrightarrow V_+^*$, we obtain

$$\langle 0| \prod_{\text{profile of } \nu} V_-(q^{\nu^t+\rho}) V_+^*(q^{-\nu-\rho})|0\rangle = \prod_{s\notin\nu} \frac{1}{1-q^{h(s)}}$$

The B-model approach in ADKMV

- Define fermion vacuum state $|vac\rangle$,

$$|vac\rangle = \prod_{m\geq 0} \psi_{m+1/2}\psi^*_{m+1/2}|0\rangle$$

The B-model approach in ADKMV

- Define fermion vacuum state $|vac\rangle$,

$$|vac\rangle = \prod_{m\geq 0} \psi_{m+1/2}\psi^*_{m+1/2}|0\rangle$$

- A Bogoliubov transformation of the fermion vacuum state from a 0-vacuum to an ∞-vacuum is defined as

$$\langle V| = \langle 0| \exp\left\{ \sum_{m,n\geq 0} a_{mn}\psi_{m+1/2}\psi^*_{n+1/2} \right\}$$

Ward identity and the information of free fermion insertions determine a_{mn}.

The B-model approach in ADKMV

- Define fermion vacuum state $|vac\rangle$,

$$|vac\rangle = \prod_{m\geq 0} \psi_{m+1/2}\psi^*_{m+1/2}|0\rangle$$

- A Bogoliubov transformation of the fermion vacuum state from a 0-vacuum to an ∞-vacuum is defined as

$$\langle V| = \langle 0| \exp\left\{\sum_{m,n\geq 0} a_{mn}\psi_{m+1/2}\psi^*_{n+1/2}\right\}$$

  Ward identity and the information of free fermion insertions determine a_{mn}.

- The topological vertex is

$$C_{\lambda\mu\nu} = \langle V|\lambda\rangle_{(1)} \otimes |\mu\rangle_{(2)} \otimes |\nu\rangle_{(3)}$$

  where λ, μ, ν are states of 2D Young diagrams.

$\mathbb{C}^3$ mirror curve description

Let (x, p) denote (u, v) or (v, w) or (w, u) on each patch respectively. And on each patch in the asympototical and core region the curve is given by

$$\text{(asym.)} \quad e^{-x} + e^{p} + 1 = 0,$$
$$\text{(core)} \quad e^{x} + e^{p} + 1 = 0.$$

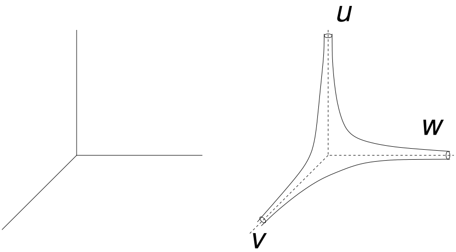


Figure : Toric diagram for $\mathbb{C}^3$ and 3 patches in the mirror curve

Symmetries

- Modular S and T transformation

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

- $\mathbb{Z}^3$ symmetry

$$ST = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad (ST)^3 = 1.$$

  $\mathbb{Z}^3$ symmetry changes $u-$, $v-$, and $w-$patches

- Symplectic structure
  $dx \wedge dp$ is preserved by S and T transformation.

Hamiltonian and D-brane probes

Let

$$z = e^x, \quad p = g_s \partial_x = g_s L_0,$$

then Hamiltonian is defined according to the curve

$$H_a = e^{-x} + q^{L_0} - 1, \quad q \equiv e^{g_s}.$$

The D-brane probes correspond to fermions inserted on the mirror curve. They satisfy the equation

$$H_a \langle e^{\varphi_+}(z) \prod_i e^{-\varphi_-}(w_i) \rangle = 0.$$

One solution gives rise to fermions's position on the curve at $\{w_i\} = \{1, q^{-1}, q^{-2}, \cdots\}$.
Similarly, the core geometry determines the position of fermions at $\{w_i'\} = \{1, q, q^2, \cdots\}$.

S-matrix and link

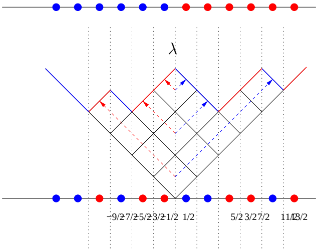
- S transformation:

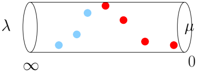
$$S : (x, p) \to (p, -x)$$

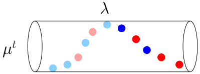
- Vertex realization

$$\langle\lambda| \prod_{\rho} V_{-}(q^{\rho}) V_{+}^{*}(q^{-\rho}) |\mu\rangle$$

$$= \quad (-1)^{|\lambda|} q^{\frac{\kappa_\mu}{2}} q^{\frac{||\lambda||}{2}} \langle\mu^t| \prod_{\text{profile of } \lambda} V_{-}(q^{\lambda^t+\rho}) V_{+}^{*}(q^{-\lambda-\rho}) |0\rangle$$







T-matrix and cut-and-join

- T transformation:

$$T : (u, v) \to (u + v, v)$$

- Field realization:

$$\varphi(u) \to \varphi(u) + g_s \partial_u \varphi(u)$$

- T transformation forms a W_0^3 symmetry

$$W_0^3 = \oint du (\partial\varphi)^3(u) \sim \frac{1}{2} \sum_{r \in \mathbb{Z}_{>0} - \frac{1}{2}} \left(r^2 + \frac{1}{12} \right) (\psi_{-r}\psi_r^* - \psi_{-r}^*\psi_r) \, .$$

Trivalent

- Result:

$$\begin{aligned}\langle \lambda, \nu, \mu \rangle &\equiv (-)^{|\nu|} q^{\frac{||\nu||}{2}} \langle \lambda | \prod_{\text{profile } \nu} V_{-}(q^{\nu^t+\rho}) V_{+}^{*}(q^{-\nu-\rho}) | \mu \rangle \\ &= s_{\nu}(q^{\rho}) \sum_{\eta} s_{\lambda/\eta}(q^{\nu^t+\rho}) s_{\mu/\eta}(q^{\nu+\rho}) \\ &= q^{\kappa_{\mu}/2} C(\mu, \lambda^t, \nu).\end{aligned}$$

Trivalent

- Result:

$$\begin{aligned} \langle \lambda, \nu, \mu \rangle &\equiv (-)^{|\nu|} q^{\frac{||\nu||}{2}} \langle \lambda | \prod_{\text{profile } \nu} V_-(q^{\nu^t+\rho}) V_+^*(q^{-\nu-\rho}) | \mu \rangle \\ &= s_\nu(q^\rho) \sum_\eta s_{\lambda/\eta}(q^{\nu^t+\rho}) s_{\mu/\eta}(q^{\nu+\rho}) \\ &= q^{\kappa_\mu/2} C(\mu, \lambda^t, \nu). \end{aligned}$$

- and a new 3-leg identity:

$$\begin{aligned} \langle \lambda, \nu, \mu \rangle &= q^{\frac{\kappa_\mu+\kappa_\nu}{2}} \langle \mu^t, \lambda, \nu^t \rangle \\ &= q^{\frac{\kappa_\lambda+\kappa_\mu}{2}} \langle \nu, \mu^t, \lambda^t \rangle \,. \end{aligned}$$

Trivalent

- Result:

$$\begin{aligned}\langle \lambda, \nu, \mu \rangle &\equiv (-)^{|\nu|} q^{\frac{||\nu||}{2}} \langle \lambda | \prod_{\text{profile } \nu} V_-(q^{\nu^t+\rho}) V_+^*(q^{-\nu-\rho}) | \mu \rangle \\ &= s_\nu(q^\rho) \sum_\eta s_{\lambda/\eta}(q^{\nu^t+\rho}) s_{\mu/\eta}(q^{\nu+\rho}) \\ &= q^{\kappa_\mu/2} C(\mu, \lambda^t, \nu).\end{aligned}$$

- and a new 3-leg identity:

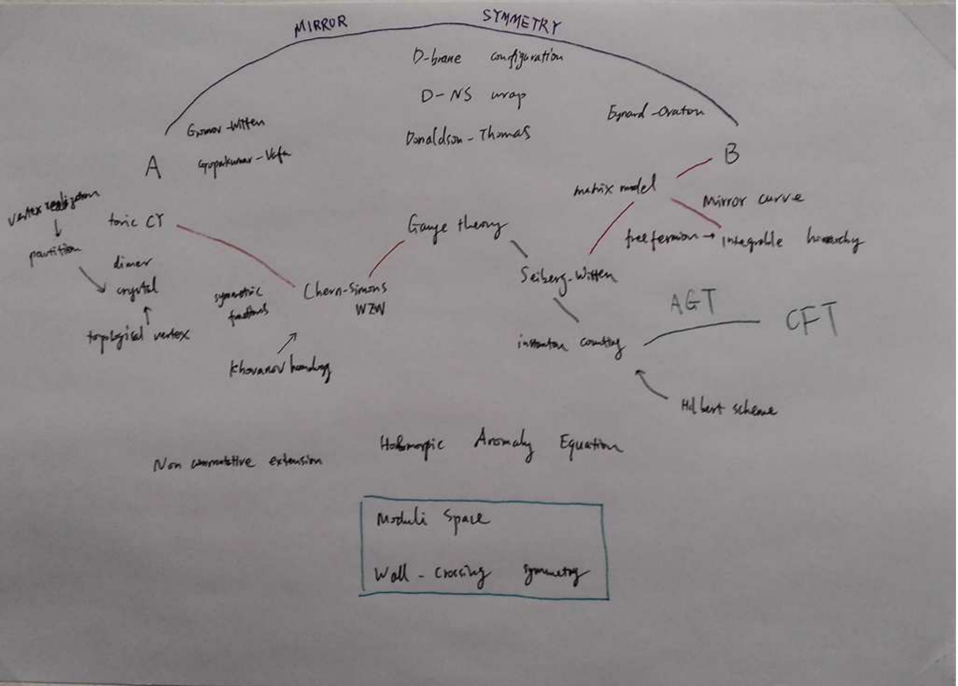
$$\begin{aligned}\langle \lambda, \nu, \mu \rangle &= q^{\frac{\kappa_\mu+\kappa_\nu}{2}} \langle \mu^t, \lambda, \nu^t \rangle \\ &= q^{\frac{\kappa_\lambda+\kappa_\mu}{2}} \langle \nu, \mu^t, \lambda^t \rangle \,.\end{aligned}$$

Reduce to Zhou's identity and have cyclic symmetry

Outline

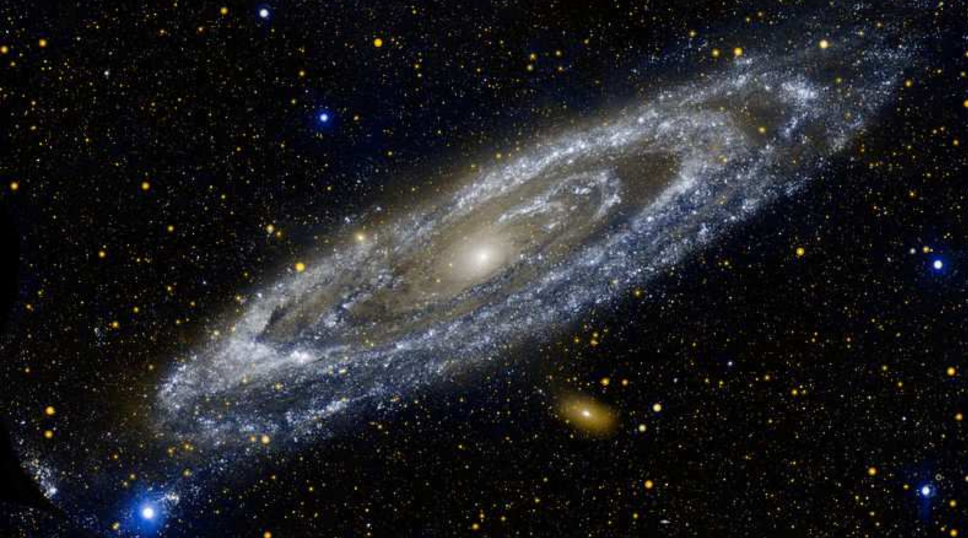
Some projects

- Refinement go
- Integrable systems
- Wall-crossing



MIRROR SYMMETRY
D-brane configuration
D-NS wrap
Donaldson-Thomas
Eynard-Orantin
Gromov-Witten
Gopakumar-Vafa
A
B
matrix model
Mirror curve
toric CY
partition
dimer
crystal
topological vertex
symmetric functions
Khovanov homology
Chern-Simons
WZW
Gauge theory
Seiberg-Witten
free fermion → integrable hierarchy
AGT
CFT
instanton counting
Hilbert scheme
Holomorphic Anomaly Equation
Non commutative extension
Moduli Space
Wall-crossing symmetry

THANK YOU

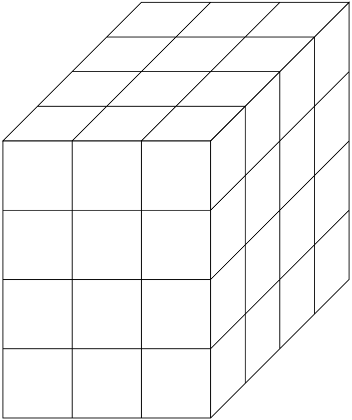


Outline

3D Young diagram partition function

A statistical cubic crystal model, namely, 3D Young diagram has partition function

$$\sum_{\pi} q^{|\pi|} = 1 + q + 3q^2 + 6q^3 + 13q^4 + \cdots$$



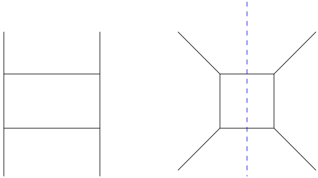
Back to 3DYoung.

Refinement

- MacMahon function

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} \to \prod_{i,j=1}^{\infty} \frac{1}{1-q^i t^{j-1}}$$

- Refined topological vertex and refined topological string



- Refined Chern-Simons theory
- Symmetric functions
  Hall-Littlewood $t \to 0$ goes to Schur
  Macdonald $q = t$ goes to Schur
- Adding parameter to WZW

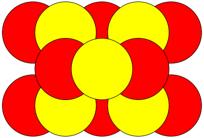
Integrable systems

- Method 1: Calogero-Sutherland (c.f. Jianfeng Wu & Ming Yu)
  Schematical way to write a Hamiltonian → bosonization and fermionization → states which are Jack symmetric functions.
- Method 2: Jimbo-Miwa and the generalization
  Phase model → $L(\lambda)$ matrix in quantum integrable model → monodromy matrix
  $T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$
  → Hamiltonian of phase model and states $B(x_1)\cdots B(x_n)\Omega$

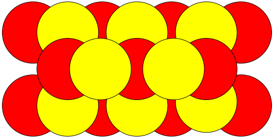
$$\phi : B(x_1)\cdots B(x_n)\Omega \to V_-(x_1)\cdots V_-(x_n)|0\rangle.$$

Wall-crossing

- Conifold



n=1



n=2

- Kontsevich-Soibelman's wall crossing formula

$$T_\gamma = \exp\{-\sum_{n=1}^{\infty} \frac{e_{n\gamma}}{n^2}\}$$

satisfies $T_{0,1}T_{1,0} = T_{1,0}T_{1,1}T_{0,1}$. Its quantum version

$$\hat{T}_\gamma = \exp\{-\sum_{n=1}^{\infty} \frac{\hat{e}_{n\gamma}}{n[n]}\}$$

satisfies $\hat{T}_{0,1}\hat{T}_{1,0} = \hat{T}_{1,0}\hat{T}_{1,1}\hat{T}_{0,1}$.

Back to back.