Time-like pion form factor in lattice QCD

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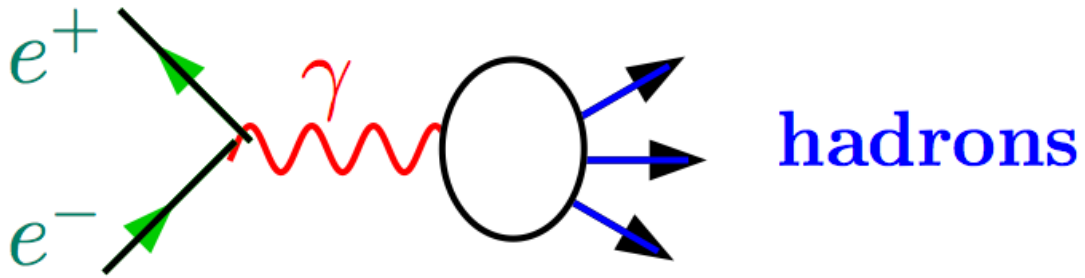
What is time-like pion form factor

Time-like pion form factor: $F_\pi(s)$

- $\gamma^* \to J_\mu \to \text{hadron}$, J_μ: hadronic vector current, $u, d, s \cdots$
  - focus on isovector part $J_\mu = J_\mu^{u-d}$



- below inelastic scattering threshold, hadrons $= \pi^+\pi^-$
- time-like pion form factor is defined by

$$\begin{aligned} \langle 0|J_\mu|\pi(p_1)\pi(p_2)\text{ in}\rangle &= -\langle \pi(p_1)\pi(p_2)\text{ out}|J_\mu|0\rangle \\ &= \sqrt{2}i(p_1-p_2)_\mu F_\pi(s), \quad s=(p_1+p_2)^2 \end{aligned}$$

- $F_\pi(s)$ describes E.M. structure of $\pi \to$
  shows how pion different from a point-like particle
  - cross section σ are measured in experiment

$$\sigma(e^+e^- \to \pi^+\pi^-) = \sigma^0(e^+e^- \to \pi^+\pi^-)|F_\pi(s)|^2$$

  - σ^0: assuming $\pi^\pm$ is point-like, tree-level cross section

Property of $F_\pi(s)$

- $F_\pi(s)$ is analytical except for the brach cut $4m_\pi^2 < s < \infty$
- in the time-like region, $s > 4m_\pi^2$, $F_\pi(s + i\epsilon)$ is complex
  - Watson's final-state theorem

$$\mathrm{Arg}[F_\pi(s + i\epsilon)] = \delta_1^1(s), \quad 4m_\pi^2 < s < \Lambda_{\mathrm{in}}^2$$

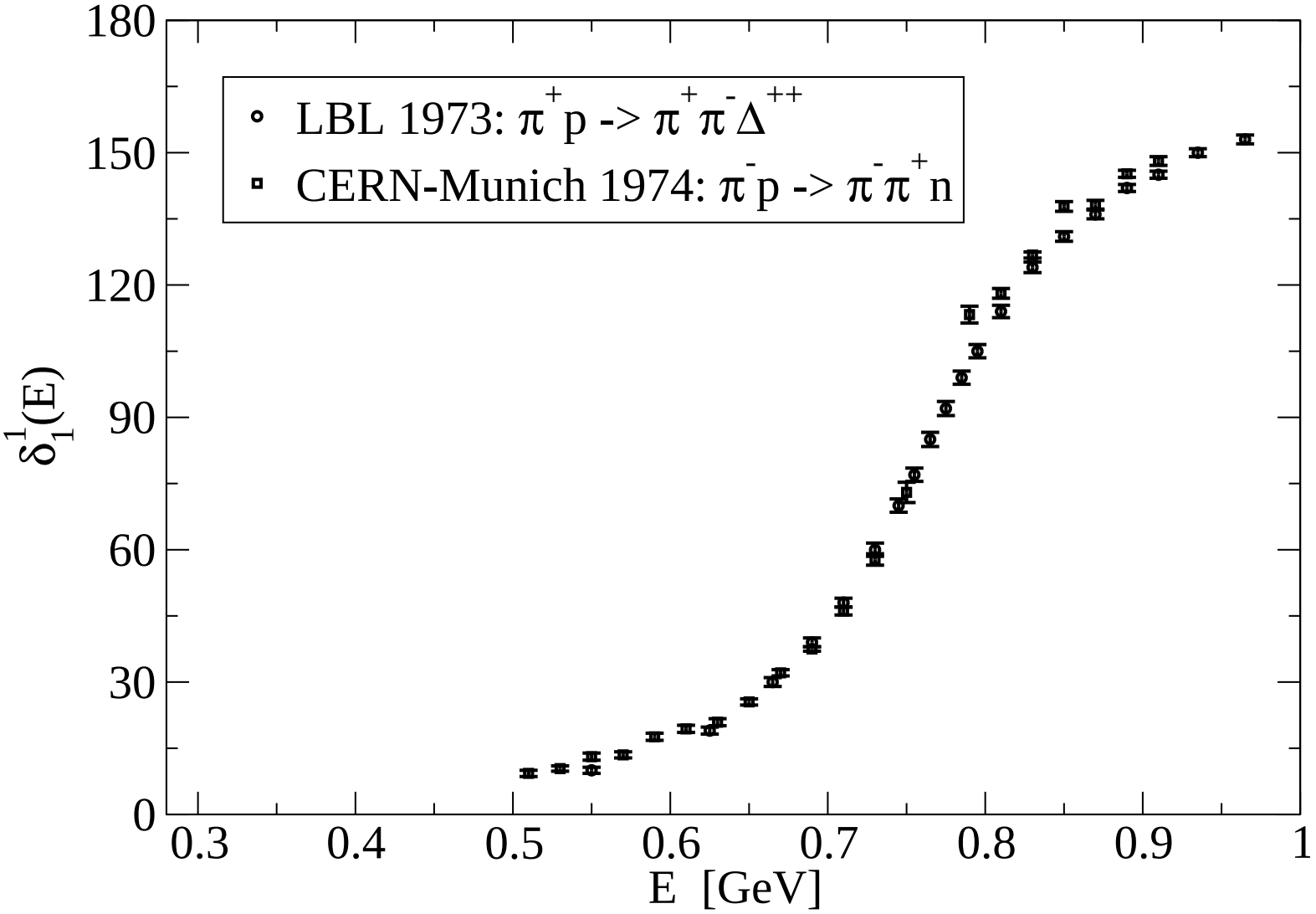
    $\delta_1^1(s)$ is $I = 1$ channel P-wave $\pi\pi$ scattering phase

  - $|F_\pi(s)|$ measured directly from experiment

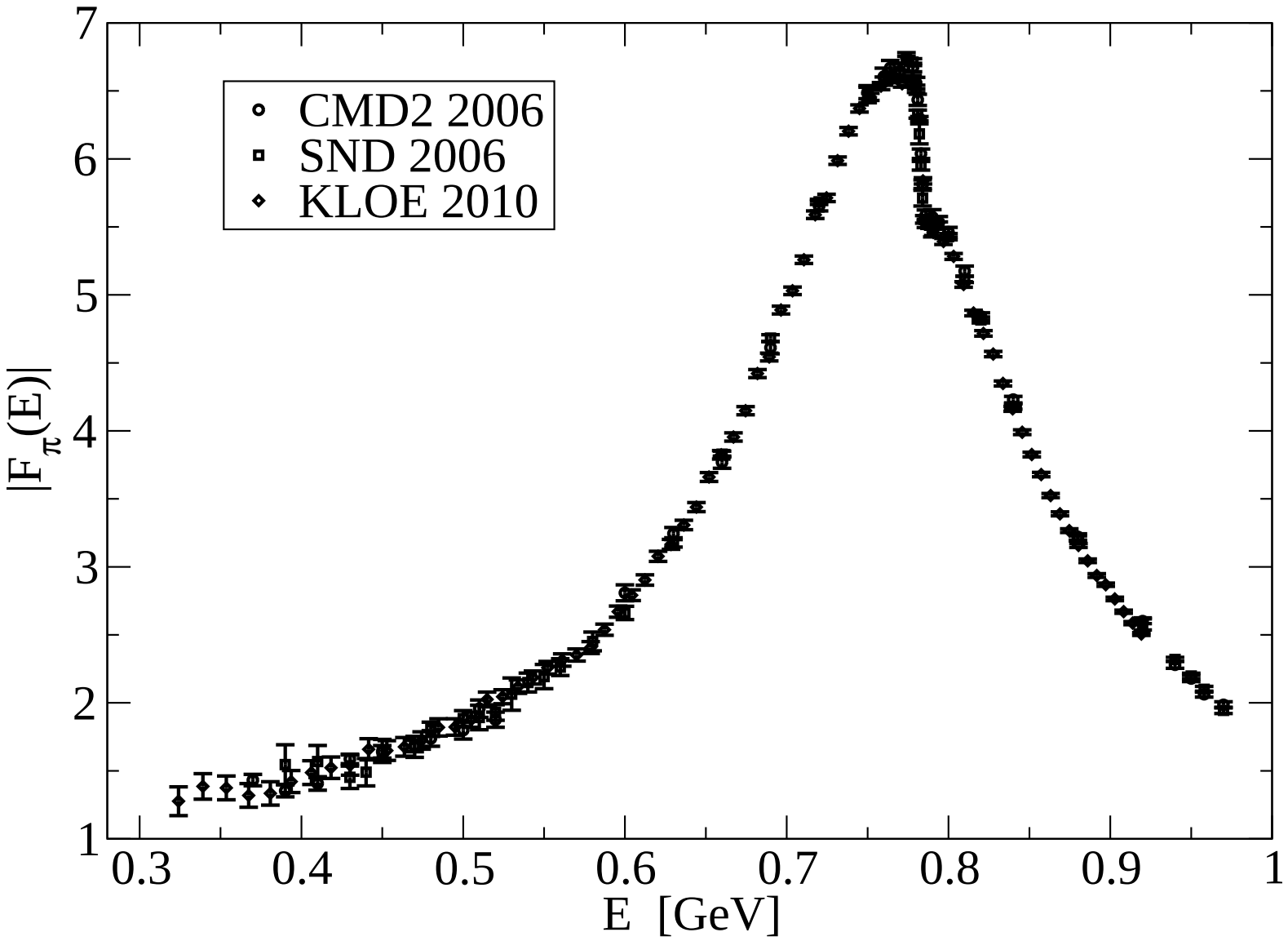
$$\sigma(e^+e^- \to \pi^+\pi^-) = \sigma^0(e^+e^- \to \pi^+\pi^-)|F_\pi(s)|^2$$

- in the space-like region, $s = q^2 < 0$, $F_\pi(q^2)$ is real

Experimental measurement of δ for $I = 1$ channel



Experimental measurement of δ for $I = 1$ channel



Why **time-like** pion form factor

Conventional pion form factor

- space-like pion form factor calculated by many groups
  - correlation functions are the primary objects in Lattice QCD
  - Euclidean time $t \to it$, operator $O(t) = e^{Ht} O(0) e^{-Ht}$
  - single pion correlation function: $\pi(t) = \sum_{\mathbf{x}} \bar{u}\gamma_5 d(\mathbf{x}, t)$

$$C_\pi(t) = \langle 0|\pi^\dagger(t)\pi(0)|0\rangle = \sum_n e^{-E_\pi^n t} |\langle n|\pi|0\rangle|^2$$

  - large t, only ground state $|n\rangle = |\pi\rangle$ saturates
  - 3-point function: $C_{\pi J \pi}(t_1, t_J, t_2) = \langle \pi(t_1) J_\mu(t_J) \pi(t_2)\rangle$

$$C_{\pi J \pi}(t_1, t_J, t_2) \xrightarrow{t_1 \gg t_J \gg t_2} \langle \pi(p_1)|J_\mu|\pi(p_2)\rangle \to F_\pi(q^2)$$

  - space-like form factor $q^2 = (p_1 - p_2)^2 < 0$

Technical difficulty in time-like form factor

- space-like form factor $\langle \pi(p_1)|J_\mu|\pi(p_2)\rangle \to F_\pi(q^2)$, $q^2 < 0$
  - indicates a process of $\pi^+ \to \gamma^* \pi^+$
- time-like form factor $\langle \pi^+(p_1)\pi^-(p_2)|J_\mu|0\rangle \to F_\pi(q^2)$, $q^2 > 4m_\pi^2$
  - indicates a process of $\gamma^* \to \pi^+\pi^-$
  - final state contains multi-hadrons
- Maiani-Testa no-go theorem [91]:
  - LQCD can't explore time-like region involving multi-hadrons

Maiani-Testa no-go theorem

- lattice results $\rightarrow$ physical ones, requiring lattice size $L \rightarrow \infty$
- for a large, quasi-infinite volume, $p_{1,2} = \pm 2\pi/L \rightarrow 0$
- ground state dominance at large t leads to

$$|\pi(\vec{p}_1)\pi(\vec{p}_2)\rangle \Rightarrow |\pi(\vec{0})\pi(\vec{0})\rangle \Rightarrow F_\pi(s = 4m_\pi^2)$$

- dense discrete energy level $\rightarrow$ hard to extract excited state
- no information of $F_\pi(s)$ can be extracted above $s > 4m_\pi^2$
- conclusion
  - we must work in the finite volume
  - finite-size correction must be treated properly

Time-like pion form factor and its applications

- from time-like pion form factor to other observables
  - $\gamma^* \to J_\mu^{u-d} \to \rho^0 \to \pi^+\pi^-$, $\langle \pi\pi | J_\mu^{u-d} | 0 \rangle \to F_\pi(s)$
  - $\gamma^* \to J_\mu^{s} \to \phi \to K^+K^-$, $\langle KK | J_\mu^{s} | 0 \rangle \to F_K(s)$
  - pion scalar form factor $\sigma \to \pi\pi$, $\langle \pi\pi | \bar{\psi}\psi | 0 \rangle \to F_\pi^s(s)$
  - $\tau \to K\pi\nu_\tau$, $W^* \to J_\mu^{sd} \to K^* \to K\pi$, $\langle K\pi | J_\mu^{sd} | 0 \rangle \to F_{K\pi}(s)$
  - nucleon form factor, $e^+e^- \to \gamma^* \to p\bar{p}$, $\langle p\bar{p} | J_\mu | 0 \rangle \to F_p(s)$
  - $\cdots$
- most of the hadrons, ρ, ϕ, σ, K^*, Δ, $\cdots$, are resonances
- $F_\pi(s)$ pave the way towards the general cases for resonances $\to$ multi-hadron

Muon g-2

- muon magnetic moment $\vec{\mu}$ is related to its spin $\vec{S}$

$$\vec{\mu} = g_\mu \frac{e\hbar}{2m_\mu c}\vec{S}$$

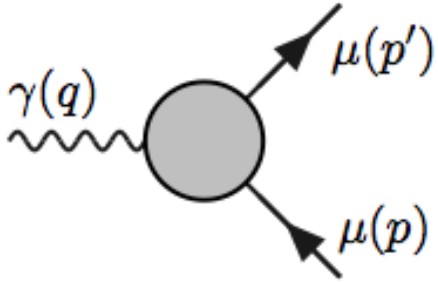
- Dirac theory predicts

$$g_\mu = 2$$

- relativistic quantum fluctuation $\Rightarrow$ g_μ deviates from 2

$$a_\mu = (g_\mu - 2)/2$$

- a_μ extracted from vertex function $\langle\mu|j^\mu_{\rm em}|\mu\rangle$: $a_\mu = F_2(0)$



$$= (-ie)\,\bar{u}(p')\Gamma_\mu(q)u(p)$$

$$= (-ie)\,\bar{u}(p')\left[\gamma^\mu F_1(q^2) + i\frac{\sigma^{\mu\nu}q_\nu}{2m_\mu}F_2(q^2)\right]u(p)\,,$$

Essentials of a_μ

- experiment at BNL [Muon g-2, PRD73:072003, 2006]

$$a_\mu^{\mathrm{EX}} = 11\,659\,208.9(6.3) \times 10^{-10}\ [0.54\,\mathrm{ppm}]$$

- Standard Model (e^+e^--based) [Davier, arXiv:0908.4300]

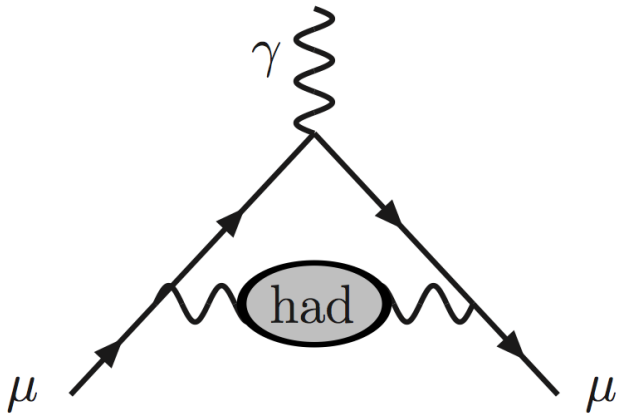
$$a_\mu^{\mathrm{SM}} = 11\,659\,183.4(4.9) \times 10^{-10}\ [0.42\,\mathrm{ppm}]$$

- discrepancy between theory and experiment

$$a_\mu^{\mathrm{EX}} - a_\mu^{\mathrm{SM}} = 25.5(8.0) \times 10^{-10}\ [3.2\,\sigma]$$

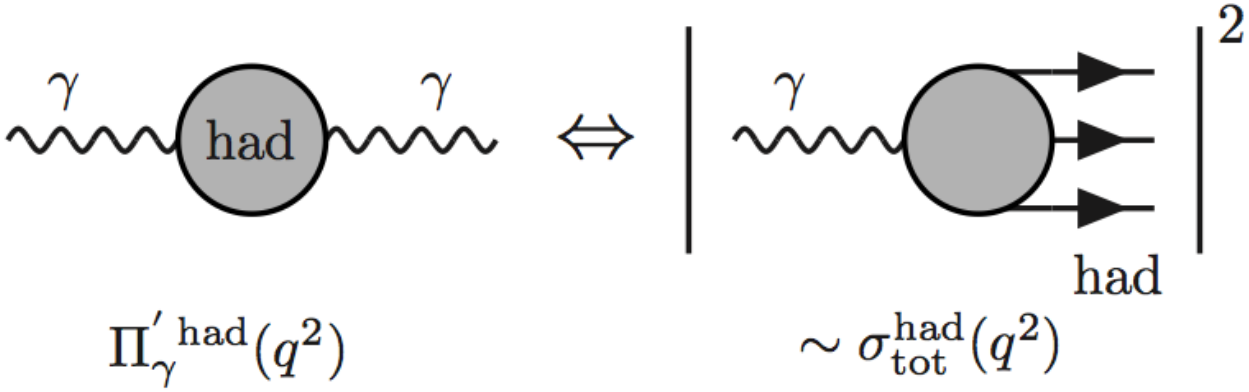
could be a hint for new physics

- 60% of theory error from hadronic vacuum polarization



$a_\mu(\pi^+\pi^-)$

- optical theorem



- $q^2 < \Lambda^2_{\mathrm{in}}$: $\sigma^{\mathrm{had}}_{\mathrm{tot}}(q^2)$ is given by $F_\pi(q^2)$

- using $F_\pi(s)$ to determine the muon g-2

$$a_\mu(\pi^+\pi^-) = 1/\pi \int_{4m_\pi^2}^{\infty} ds\, w(s)|F_\pi(s)|^2$$

where $w(s)$ is a known weight function

How to calculate time-like $F_\pi(s)$

$F_\pi(s)$: complex angle

- enclose $\pi\pi$ in a box with a size $L \sim$ a few fermi
- two-pion correlation function

$$C_{\pi\pi}(t) = \langle 0|(\pi\pi)^\dagger(t)(\pi\pi)(0)|0\rangle = \sum_n e^{-E^n_{\pi\pi}t}|\langle \pi\pi, n|\pi\pi|0\rangle|^2$$

- discrete $E^n_{\pi\pi}$ are what we extract from lattice calculation
- turn off the interactions, two pions are free

$$E^{\text{free}}_{\pi\pi} = \sqrt{m_\pi^2 + \vec{p}_1^2} + \sqrt{m_\pi^2 + \vec{p}_2^2}\,, \quad \vec{p}_{1,2} = \frac{2\pi}{L}\vec{n}_{1,2}$$

- difference $\triangle E = E^n_{\pi\pi} - E^{\text{free}}_{\pi\pi}$ arises from interactions H_{int}
- scattering phase δ is determined by the interactions H_{int}
- arising from same H_{int}, relation between $E^n_{\pi\pi}$ and δ exists?
- Lüscher established a formula to relate $E_{\pi\pi}$ to $\delta(s)$

$$n\pi - \delta(k) = \phi(q), \quad n \in \mathbb{Z}, \quad q = \frac{k}{2\pi/L}, \quad E_{\pi\pi} = 2\sqrt{m_\pi^2 + k^2}$$

$\phi(q)$ is a known function, irrelevant to interaction details

$F_\pi(s)$: modulus

- in the $I = 1$ channel, J_μ has same quantum number as $\pi\pi$

$$\int_V d^3x \, \langle 0|J_\mu(x,t)J_\mu(0)|0\rangle_V = \sum_n |\langle \pi\pi, n|J_\mu|0\rangle_V|^2 e^{-E^n_{\pi\pi}t}$$

- $E_{\pi\pi}$ & $|\langle \pi\pi, n|J_\mu|0\rangle_V|^2$: quantities calculated from LQCD
  - $E_{\pi\pi}$ can be used to determine δ, the complex angle of $F_\pi(s)$
  - question: how we relate $|\langle \pi\pi, n|J_\mu|0\rangle_V|^2$ to $|F_\pi(s)|^2$?
- recall Lüscher's formula

$$\frac{\phi(q) + \delta(k)}{\pi} = n$$

- $V \to \infty$, finite volume sums $\to$ infinite volume integrals [Lin, Martinelli, Sachrajda, Testa, 2001]

$$\sum_n \to \int dE \, \rho_V(E), \quad \rho_V(E) = \frac{dn}{dE} = \frac{q\phi'(q) + k\delta'(k)}{4\pi k^2} E$$

$\rho_V(E)$ can be viewed as the density of states at energy E

$F_\pi(s)$: modulus

- finite volume

$$\int_V d^3x\, \langle 0|J_\mu(x,t)J_\mu(0)|0\rangle_V \xrightarrow{V\to\infty} \int dE\, \rho_V(E)|\langle \pi\pi, E|J_\mu|0\rangle_V|^2 e^{-Et}$$

- infinite volume: spectral representation for vector correlator

$$C_\infty(t) = \int d^3x\, \langle 0|J_\mu(x,t)J_\mu(0)|0\rangle = \int_{2m_\pi}^\infty dE\, f(E)e^{-Et}$$

$$f(E) = \frac{1}{6\pi^2}\frac{k^3}{E}|F_\pi(s)|^2, \quad s = E^2 = 4(m_\pi^2 + k^2)$$

- final relation

$$|F_\pi(s)|^2 = \frac{3\pi s}{2k^5}\left(q\frac{\partial\phi(q)}{\partial q} + k\frac{\partial\delta(k)}{\partial k}\right)|\langle \pi\pi, E|J_\mu|0\rangle_V|^2$$

- result correct, but demonstration not very satisfactory
  - equivalent integral does not mean equivalent integrand
  - integral $\int_{2m_\pi}^\infty$ runs over inelastic region
- indirect demonstration by introducing vector boson $W \to \pi\pi$ [Lellouch, Lüscher, 2000; Meyer, PRL 2011]
- direct demonstration, my work at Columbia with N. Christ

Play with the method

Lattice setup

- $N_f = 2 + 1$ overlap fermion, from JLQCD collaboration
- $m_\pi = 380$ and 290 MeV, both with a lattice size $L = 2.6$ fm
- vector current operator: $\bar{\psi}$-ψ fields or $\pi^+\pi^-$ meson pairs

$$
\begin{aligned}
J_{\vec{\mu}}(\vec{P}, t) &= \left(\bar{\psi}(\vec{\mu}\cdot\vec{\gamma})\frac{\tau}{2}\psi\right)(\vec{P}, t) \\
(\pi\pi)(t) &= \frac{1}{2}(\pi^+(\vec{p}_1, t+1)\pi^-(\vec{p}_2, t-1) + \pi^+(\vec{p}_1, t-1)\pi^-(\vec{p}_2, t+1)) \\
&- \frac{1}{2}(\pi^+(\vec{p}_2, t+1)\pi^-(\vec{p}_1, t-1) + \pi^+(\vec{p}_2, t-1)\pi^-(\vec{p}_1, t+1)) \\
&\Rightarrow J^{\pi\pi}_{\vec{\mu}}(\vec{P}, t), \quad \vec{P} = \vec{p}_1 + \vec{p}_2, \quad \vec{\mu} = \frac{\vec{p}_1 - \vec{p}_2}{|\vec{p}_1 - \vec{p}_2|}
\end{aligned}
$$

- under parity symmetry $\vec{p}_{1,2} \to \vec{p}_{2,1}$, $(\pi\pi)(t) \to -(\pi\pi)(t)$
- under isopsin symmetry $\pi^\pm \to \pi^\mp$, $(\pi\pi)(t) \to -(\pi\pi)(t)$

Spatial momentum

- spatial momentum configurations

$\pi(1,0,0)\pi(-1,0,0)$	$J_{\vec{\mu}}(0,0,0)$	circle in the plot
$\pi(1,0,0)\pi(0,0,0)$	$J_{\vec{\mu}}(1,0,0)$	square
$\pi(1,1,0)\pi(0,0,0)$	$J_{\vec{\mu}}(1,1,0)$	diamond
$\pi(1,1,1)\pi(0,0,0)$	$J_{\vec{\mu}}(1,1,1)$	triangle up
$\pi(1,0,0)\pi(0,1,0)$	$J_{\vec{\mu}}(1,1,0)$	triangle left

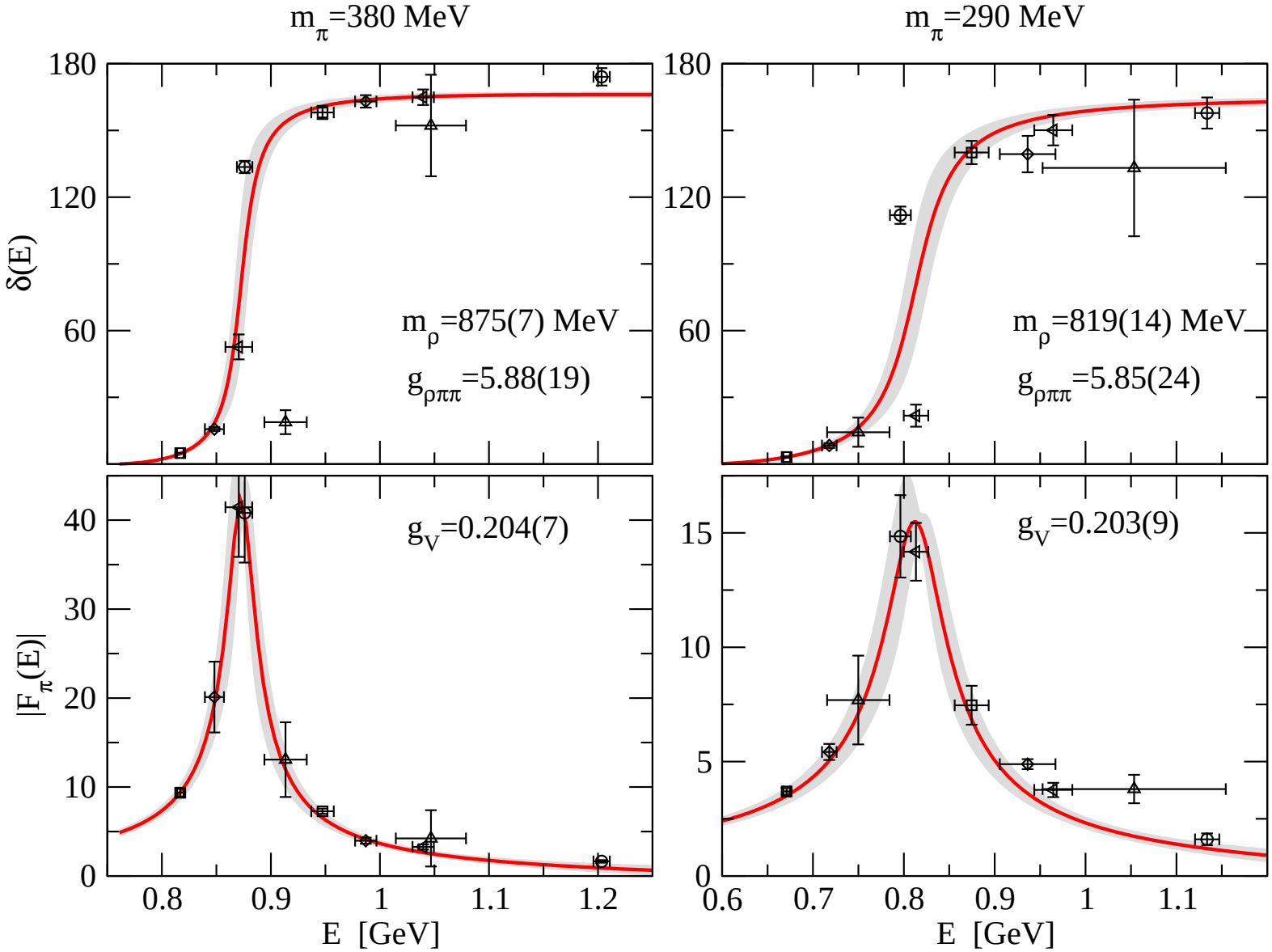
- all-to-all propagagtor technique to construct the correlator

$$\left\langle [(\pi\pi)(t)]^\dagger (\pi\pi)(0) \right\rangle, \left\langle [(\pi\pi)(t)]^\dagger J(0) \right\rangle, \left\langle [J(t)]^\dagger (\pi\pi)(0) \right\rangle, \left\langle [J(t)]^\dagger J(0) \right\rangle$$

- variational method to calculate $E_{\pi\pi}$ & $|\langle \pi\pi, E|J_\mu|0\rangle_V|^2$ for lowest two states
- 5 moving frames $\times$ 2 states $=$ 10 data points
- $E_{\pi\pi} \to \delta_1^1, \quad |\langle \pi\pi, E|J_\mu|0\rangle_V|^2 \to |F_\pi(s)|^2$

Preliminary results

- $N_f = 2 + 1$ overlap fermion, $m_\pi = 380$ and 290 MeV, $L = 2.6$ fm



Modeling the data

Fit strategy

- $\gamma^* \to \pi\pi$: at low energies $\pi\pi$ scattering dominated by the production and decay of ρ-meson
- introducing ρ-meson prop. $i/(s - m_\rho^2)$, VMD form gives

$$F_\pi(s) = A/(m_\rho^2 - s), \quad A = m_\rho^2$$

  normalization $F_\pi(0) = 1$ requested by charge conservation
- $\pi\pi$ branch cut effect, Gounaris and Sakurai [68] employ

$$F_\pi(s) = A/(m_\rho^2 - s - \Pi_\rho(s)), \quad A = m_\rho^2 - \Pi_\rho(0)$$

  $\Pi_\rho(s)$: ρ self-energy produced by the two-pion loop
- near resonance energy, tree-level $\rho \to \pi\pi$ amplitude

$$\langle \pi(k_1)\pi(k_2), \mathrm{out} | \rho(\varepsilon, k_0), \mathrm{in} \rangle = g_{\rho\pi\pi}\, \varepsilon_\mu \cdot (k_1 - k_2)^\mu$$

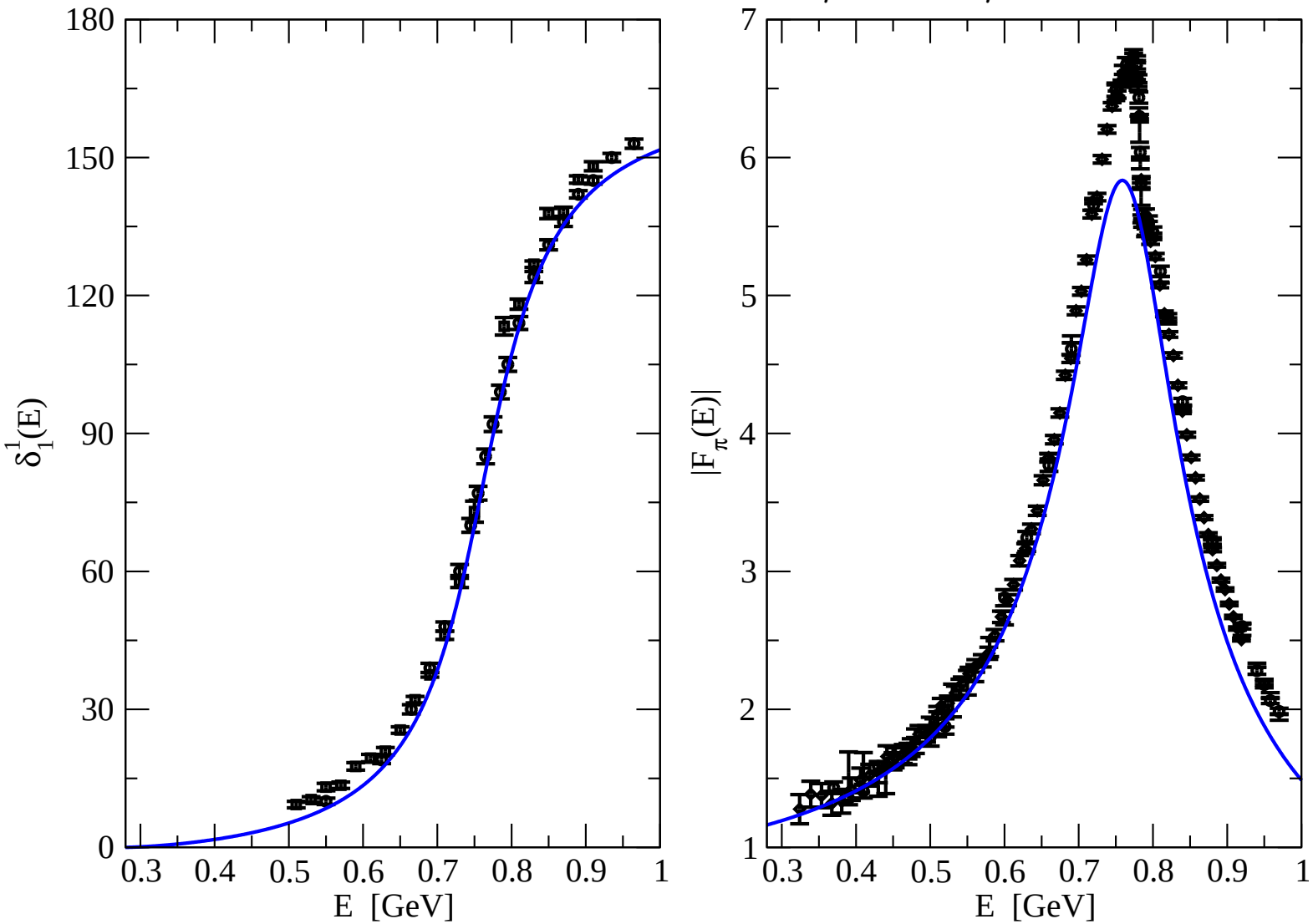
- using optical theorem we have

$$\mathrm{Im}\, \Pi_\rho(s) = \frac{g_{\rho\pi\pi}^2}{6\pi} \frac{k^3}{\sqrt{s}}, \quad k = \sqrt{s/4 - m_\pi^2}$$

- $\mathrm{Re}\, \Pi_\rho(s)$ is related to $\mathrm{Im}\, \Pi_\rho(s)$ by dispersion relation

Draw back of GS model

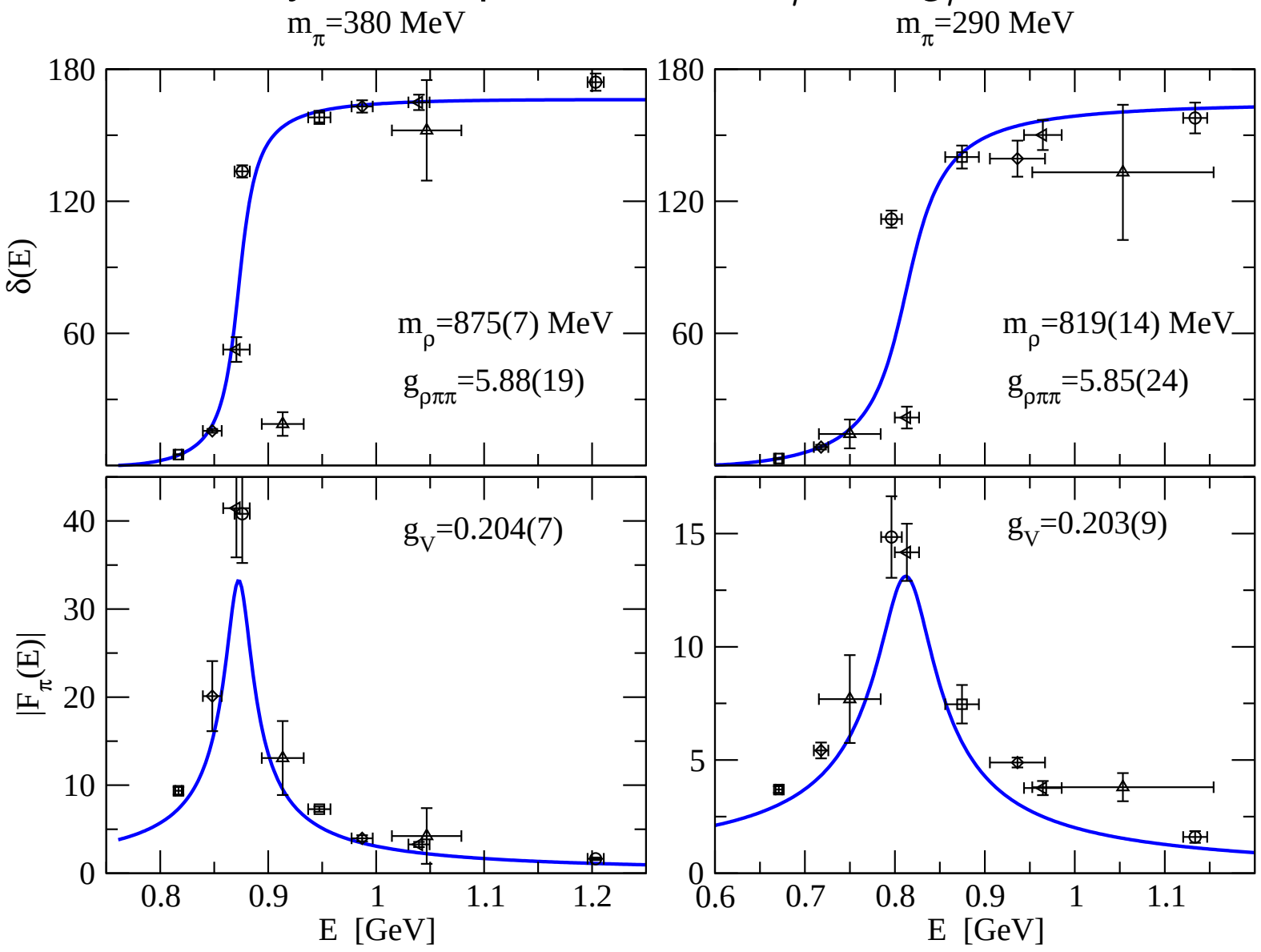
- GS model rely on two parameters: m_ρ and $g_{\rho\pi\pi}$



- GS model v.s. experimental data

Draw back of GS model

- GS model rely on two parameters: m_ρ and $g_{\rho\pi\pi}$



- GS model v.s. lattice data

Modification to GS model

- GS form gives too small value for $|F_\pi(s)|$ at $s \approx m_\rho^2$

$$F_\pi(s) = A/(m_\rho^2 - s - \Pi_\rho(s)), \quad A = m_\rho^2 - \Pi_\rho(0)$$

- extend GS form by including $\rho(1450)$, $\rho(1700)$
  - doubts: higer resonances really affect ρ so much
- GS fails at $s \approx m_\rho^2$, where $\gamma^* \to \rho \to \pi\pi$
  - $\gamma^* \to \rho$ is related to $\rho \to \gamma^* \to l^+l^-$, $\Gamma(\rho \to l^+l^-) \to g_V$
  - $\rho \to \pi\pi$ is related to $\Gamma(\rho \to \pi\pi) \to g_{\rho\pi\pi}$
- considering γ^*-ρ mixing, near $s = m_\rho^2$, VMD gives

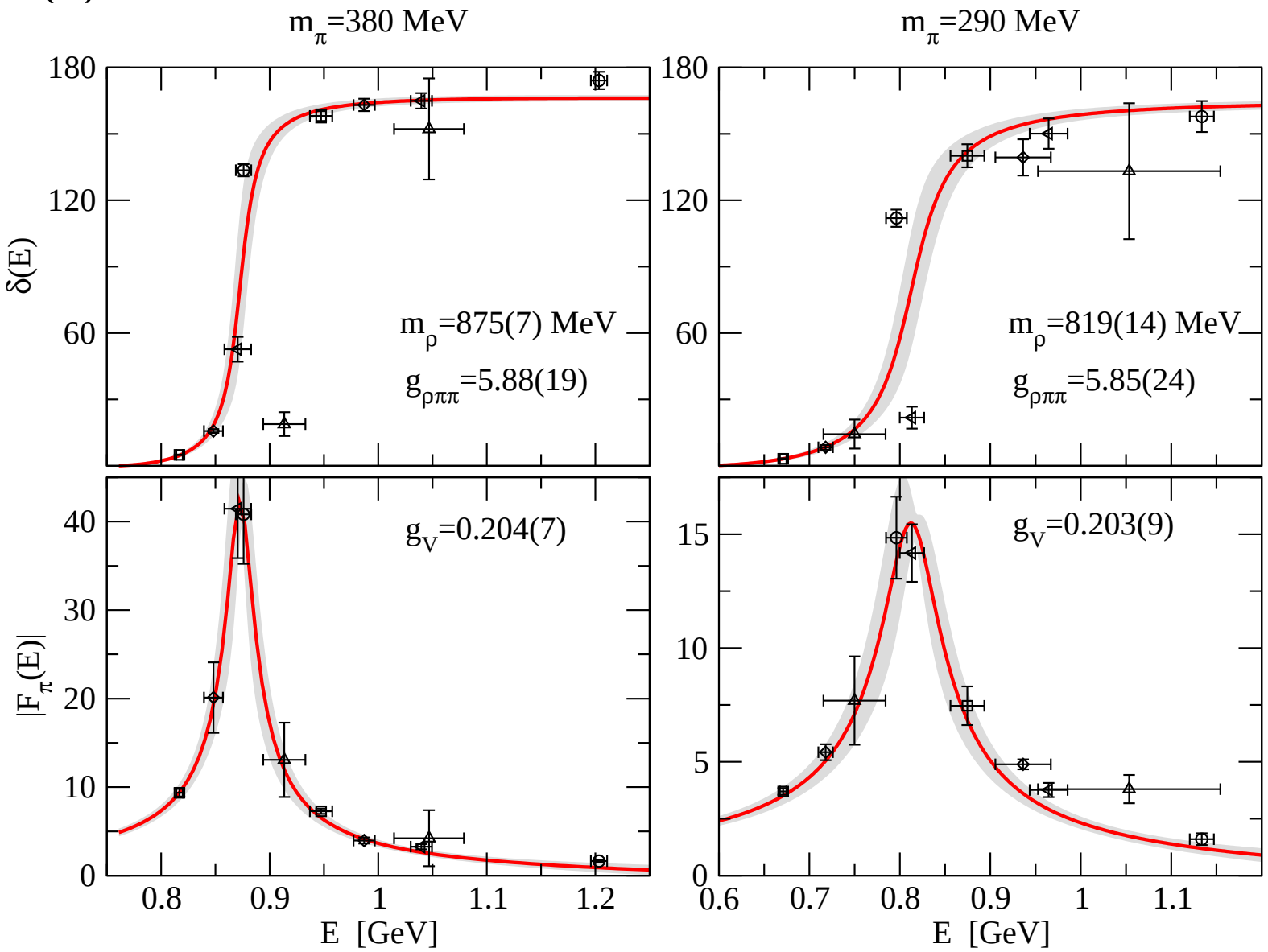
$$F_\pi(s) = \frac{g_{\rho\pi\pi} g_V m_\rho^2}{m_\rho^2 - s - \Pi_\rho(s)}, \quad F_\pi(0) \neq 0$$

- introduce a s-dependent $A(s) = a_0 + a_1 s + a_2 s^2 + \cdots$

$$A(s)\Big|_{s=0} = m_\rho^2 - \Pi_\rho(0), \quad A(s)\Big|_{s=m_\rho^2} = g_{\rho\pi\pi} g_V m_\rho^2$$

Fit the lattice data

- $A(s) = a_0 + a_1 s + a_2 s^2$: fit looks fine



- through the fit, we determine m_ρ, $g_{\rho\pi\pi}$, g_V

Conclusion

	Lattice		Experiment
m_π	380 MeV	290 MeV	140 MeV
m_ρ	875(7) MeV	819(14) MeV	775 MeV
$g_{\rho\pi\pi}$	5.88(19)	5.85(24)	5.98
g_V	0.204(7)	0.203(9)	0.202

- first lattice calculation on time-like $F_\pi(s)$
- $g_{\rho\pi\pi}$ shows a mild pion mass dependence, consistent with previous studies from ETMC and PACS-CS
- new way to determine g_V, finite-size effect considered
- future direction:
  - move to physical pion mass
  - other applications