HEE and SSA

Holographic Entanglement Entropy and Strong Subadditivity

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Outline

- AdS/CFT
- EE in QFT
- HEE
- SSA

AdS/CFT correspondence

- Conjecture: gauge/gravity duality
  CFT_d is dual to AdS_{d+1}
- An example: N=4 super-YM in 4d $\Longleftrightarrow$ AdS_5 space
- strong/weak duality,
  calculations in weakly curved gravity $\Longrightarrow$ strongly coupled QFT
  e.g. holographic entanglement entropy
  $\Longrightarrow$ EE in QFT

EE in QFT

- Entanglement entropy is a measurement tool of correlations between different subsystems of quantum system, many-body systems, or QFT
- Generally hard to measure in experiments

For a pure state system $\rho = |\Psi\rangle\langle\Psi|$

Hilbert space: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

The reduced density matrix of subsystem A: $\rho_A = \mathrm{Tr}_B(\rho)$

von Neumann entropy $S_A = -\mathrm{Tr}_A\left(\rho_A \log(\rho_A)\right)$

measuring the correlation between A and B, i.e. entanglement entropy.

For pure state, $S_{total} = 0$, while $S_A = S_B \neq 0$ in general.

thus not extensive.

If $|\Psi\rangle = |\Psi\rangle_A \otimes |\Psi\rangle_B \longrightarrow S_A = 0,$ not entangled.

Define Renyi entropy $S_A^{(n)} = \frac{1}{1-n} \log\left(\mathrm{Tr}(\rho_A)^n\right)$

also for pure QM state, it has $S_A^{(n)} = S_B^{(n)}$

And one could check, $S_A = \lim_{n\to 1} S_A^{(n)}$

- Mutual information $I(A,B) = S_A + S_B - S_{AB}$

  or in Renyi entropy $I(A,B)^{(n)} = S_A^{(n)} + S_B^{(n)} - S_{AB}^{(n)}$

  where $AB = A \cup B$

  measure correlations between A and B.

  Suppose $B = \bar{A} \longrightarrow S_{AB}^{(n)} = S_{total}^{(n)}$

  If n=1, and pure state, then

$$S_{total} = 0, \quad S_A = S_B, \quad \longrightarrow \quad I(A,B) = 2S_A.$$

- Connection with thermal entropy?

  If the total state is not pure, but mixed instead, then

$$\rho_{total} = \rho_{thermal} = \frac{e^{-\beta H}}{\mathrm{Tr}\left(e^{-\beta H}\right)}$$

  EE is the limit of vanishing B,

$$\rho_A = \lim_{B\to 0} \rho_{total} = \rho_{th} \longrightarrow S_A = \lim_{B\to 0} S_{th}$$

Properties of EE

- Subadditivity: $S_A + S_B \geq S_{AB}$
- Araki-Lieb/triangle inequality: $|S_A - S_B| \leq S_{AB}$
- Strong subadditivity(SSA):

$S_{AB} + S_{BC} \geq S_{ABC} + S_B$, or $S_A + S_B \geq S_{A\cup B} + S_{A\cap B}$

$S_{AB} + S_{BC} \geq S_A + S_C$

$\longleftrightarrow$ concavity of von Neumann entropy

e.g. If (A,B,C) take value of numbers,

and for simplicity suppose A=C,

define x=A+B+C, y=C,

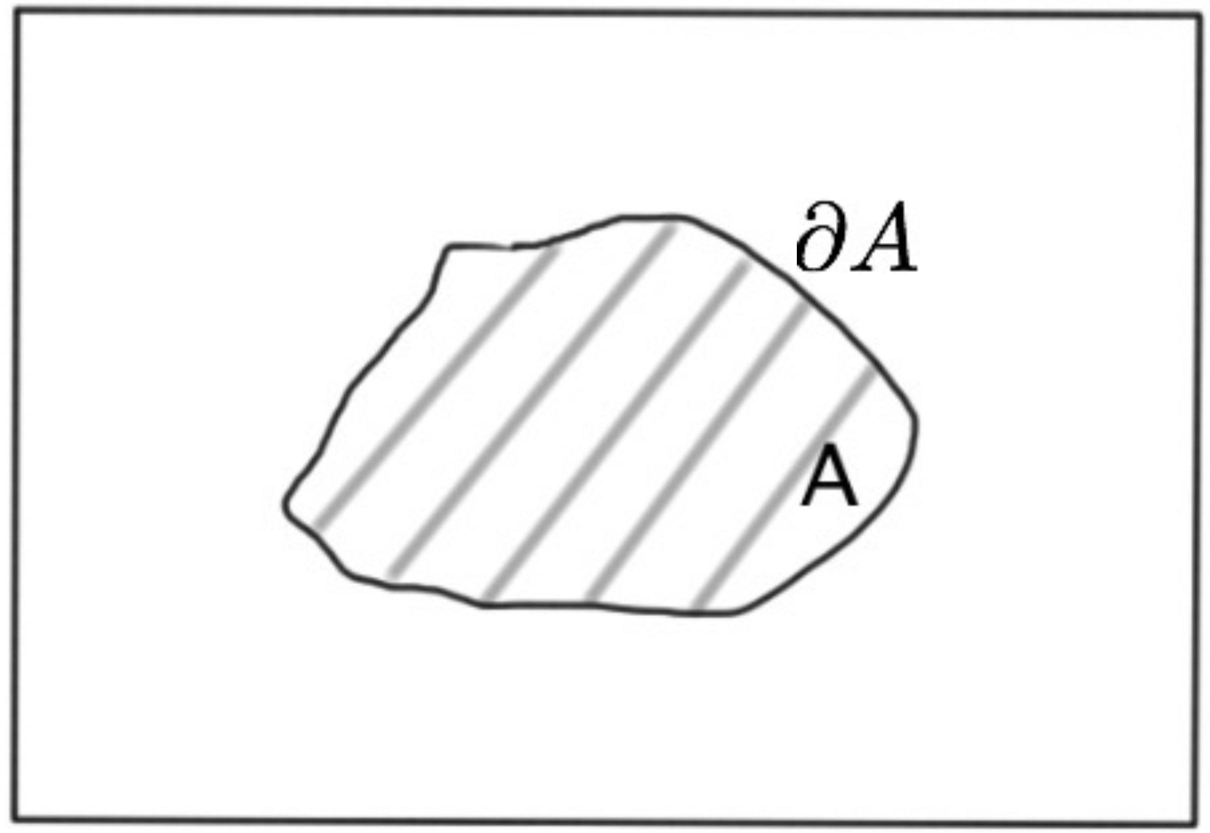
then SSA $\longleftrightarrow \quad 2S\left(\frac{x+y}{2}\right) \geq S_x + S_y \quad \to \quad \frac{d^2}{dx^2}S(x) \leq 0$

Properties of EE: Area law

In (d+1)-dims,

$$S_A \sim \frac{\mathrm{Area}(\partial A)}{a^{d-1}} + \text{subleading terms}$$

where ∂A (d-1)-dim,
and a is UV cutoff



Exceptions:

- For free bosons in (1+1)-dim, consider an interval with length l,

$$S_A = \frac{c}{3} \log\left(\frac{l}{a}\right),$$

- For fermions,

$$S_A \sim \left(\frac{l}{a}\right)^{d-1} \log\left(\frac{l}{a}\right) + \cdots,$$

Properties of EE: Area law

- Area law is violated for highly excited states, where thermal entropy plays a more important role.
- Proof only available for free field theories.
- AdS/CFT could show area law for strong coupled QFT, as long as it has UV fixed points.

EE in QFT: the calculations

Suppose λ_i are eigenvalues for ρ_A

$$S_A = -\mathrm{Tr}\left(\rho_A \log \rho_A\right) \;\;\rightarrow\;\; -\sum_i \lambda_i \log(\lambda_i)$$

Since $\sum_i \lambda_i = 1, \quad \lambda_i \in [0,1] \longrightarrow \sum \lambda_i^n \in [0,1]$ convergent, for $n \geq 1$

$\longrightarrow \quad \mathrm{Tr}\left(\rho_A\right)^n = \sum_i \lambda_i^n$ convergent and analytic,

for Re[n]≧1, n not necessarily integers

$$\longrightarrow \quad S_A = -\lim_{n\to 1} \frac{\partial}{\partial n} \mathrm{Tr}(\rho_A)^n = \lim_{n\to 1} S_A^{(n)}$$

"replica method"

Replica Method

First take n>0, in integer values, then

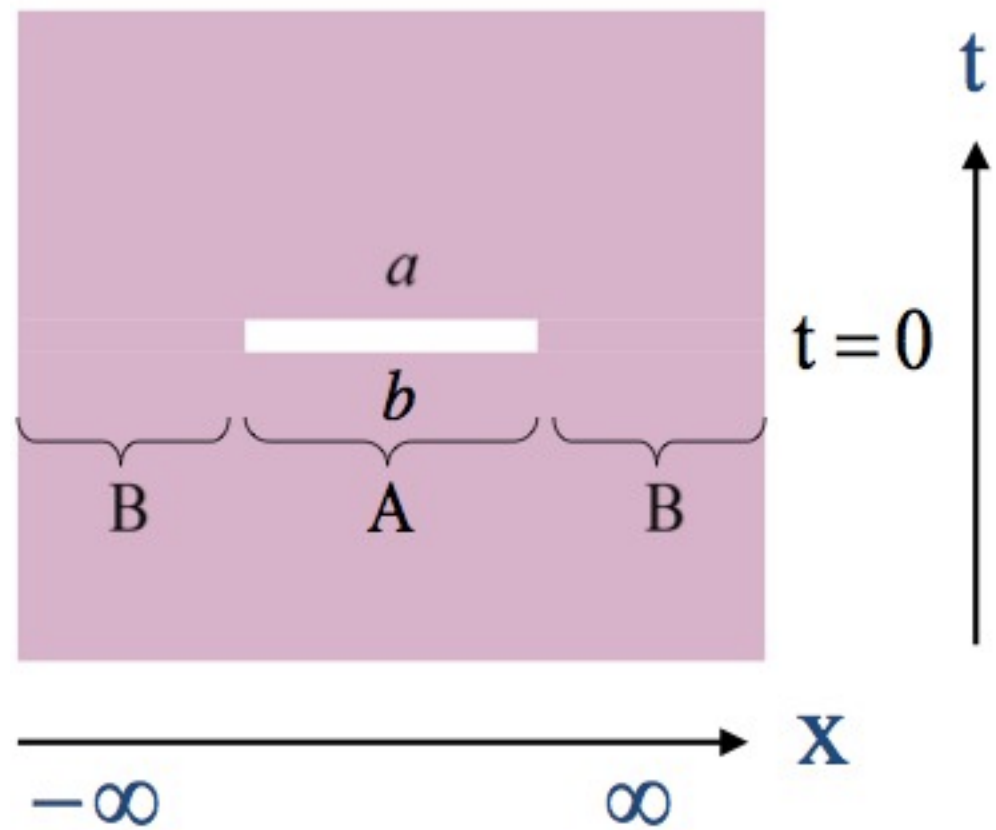
$\mathrm{Tr}(\rho_A)^n \longleftrightarrow$ partition function on Riemann surface

A simple example: (1+1)-d, scalar Φ in interval along x,

If A is an interval x∈(u,v),

$$\rho_A = \mathrm{Tr}_B|\Psi\rangle\langle\Psi|,$$

$$[\rho_A]_{ab} =$$



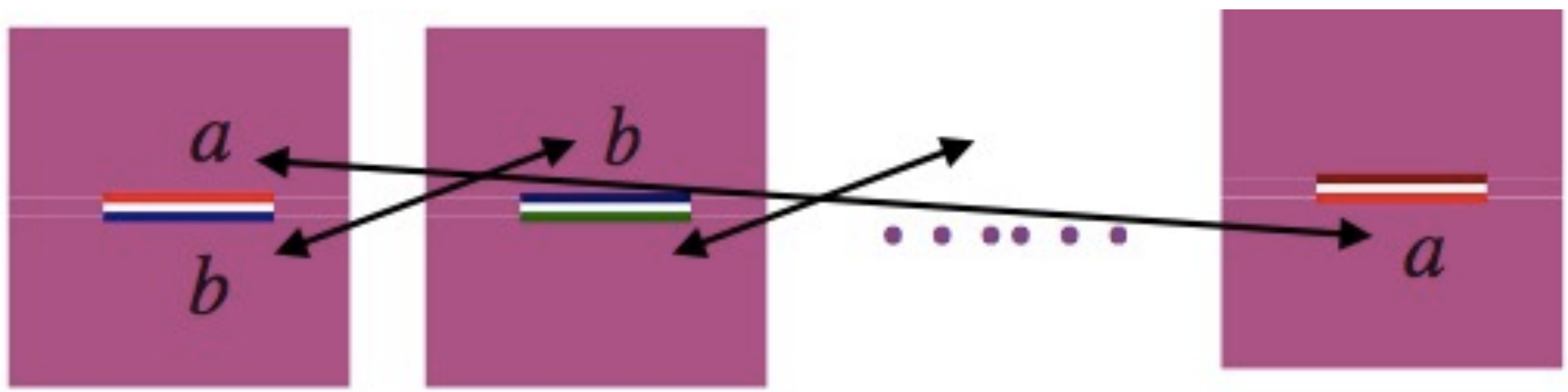
Could calculate on (x,τ=it) plane, where (a,b) =(x+iε, x-iε) are the upper and lower "boundary" of the branch cut A.

If consider finite temperature T, one has to sew τ=0 together with τ=β, forming a cylinder.

Replica Method

Trace $\mathrm{Tr}\,(\rho_A)^n = [\rho_A]_{ab}\,[\rho_A]_{bc}\cdots\cdots[\rho_A]_{ja}$

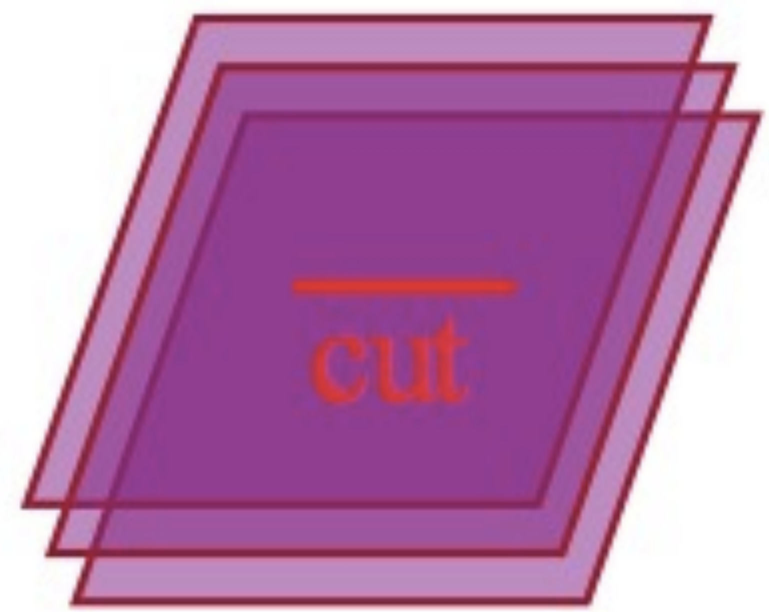
Need to glue together the "boundaries" of branch cut successively.

$\mathrm{Tr}\,(\rho_A)^n =$ 

Path integral over n-sheeted Riemann surface,
with non-trivial topological structure.

$$\mathrm{Tr}\,(\rho_A)^n = \frac{Z_n(A)}{(Z_1)^n}$$

n sheets



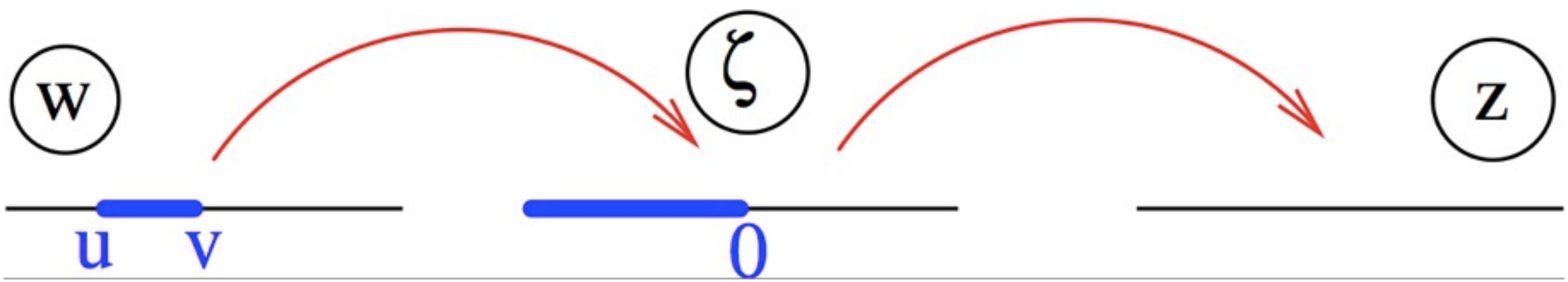
Generally, with N intervals, (u_1,v_1), (u_2,v_2),..., (u_N,v_N),
need to trace over Riemann surface $R_{n,N}$

Replica Method

Apply a series of mapping:

define $\omega = x + i\tau, \quad \bar{\omega} = x - i\tau.$

- conformal mapping, $\omega \to \xi \equiv \dfrac{\omega - u}{\omega - v}$
- reduce the sheets' number, $\xi \to z = \xi^{1/n}$



The partition function would be integrated on a complex plane z.

i.e. $R_{n,1} \longrightarrow$ z-plane.

Generally, it's not doable for multi-intervals, which correspond with much more complicated Riemann surfaces.

Replica Method

From CFT calculation, one finds

$$\frac{Z_n(A)}{(Z_1)^n} \propto \mathrm{Tr}(\rho_A)^n = c_n \left(\frac{v-u}{a}\right)^{-\frac{c}{6}\left(n-\frac{1}{n}\right)}$$

$$\longrightarrow S_A^{(n)} = \frac{c}{6}\left(1+\frac{1}{n}\right)\log\frac{l}{a} + c'_n,$$

$$S_A = \frac{c}{3}\log\frac{l}{a} + c'_1.$$

At finite temperature, $$S_A = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh\frac{\pi l}{\beta}\right) + c'_1.$$

Finite size of system, length L with periodic condition:

$$S_A = \frac{c}{3}\log\left(\frac{L}{\pi a}\sin\frac{\pi l}{L}\right) + c'_1.$$

Time evolution, QM quench

QM quench, at t=0, and evolves unitarily after.

global quench $\longrightarrow$ linearly growing

local quench $\longrightarrow$ logarithmically growing

- global quench:

  Only consider (t,l)≫microscopic length and time, i.e. RG

$$S_A \sim -\frac{c}{3}\log\tau_0 + \begin{cases} \frac{\pi c t}{6\tau_0}, & t < \frac{l}{2}, \\ \frac{\pi c l}{12\tau_0}, & t > \frac{l}{2}. \end{cases}$$

- local quench:

  No translational inv. NO RG.

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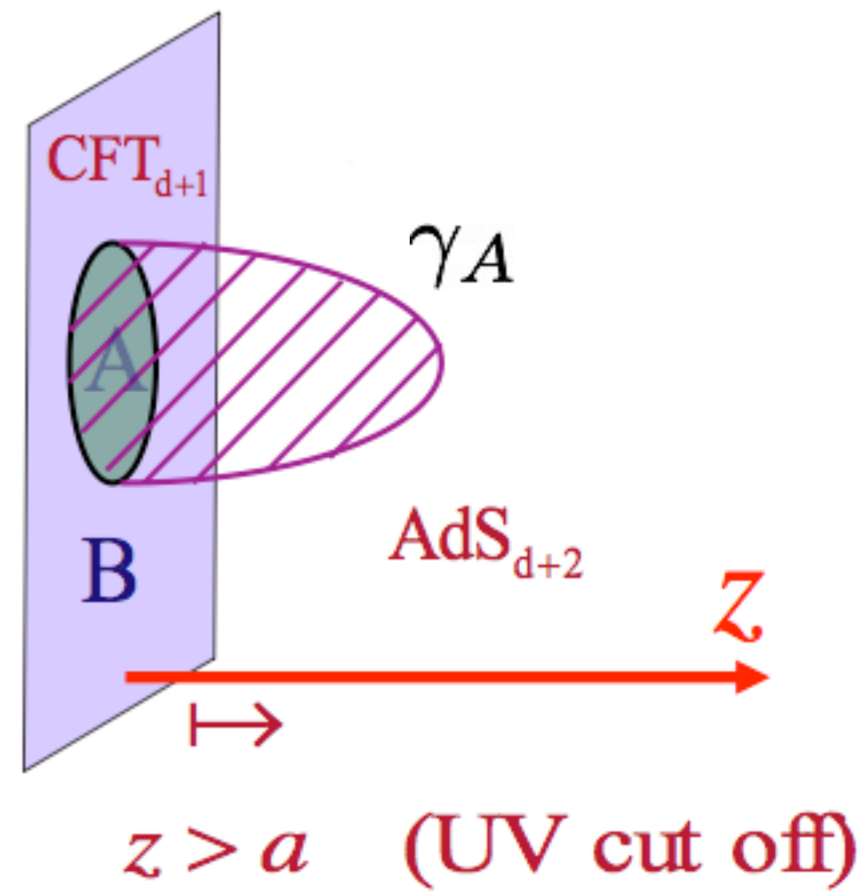
Holographic Entanglement Entropy

For a region A on AdS's boundary, find the hypersurface γ_A in AdS bulk, with the same boundary of A and with minimal area.



Dimensions:

AdS_{d+2} bulk, CFT_{d+1} on boundary,

A: (d+1), γ_A: (d+1)~A, ∂A: d

holographic EE: $S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N}$.

Ryu-Takayanagi, 06

Area law:

$$\mathrm{Area}(\gamma_A) \sim R^d \frac{\mathrm{Area}(\partial\gamma_A)}{a^{d-1}} + \text{subleading terms}$$

$$\longrightarrow \quad S_A \sim \frac{\mathrm{Area}(\partial A)}{a^{d-1}} + \cdots$$ agree with QFT

If time dependent:

minimal surface, RT entropy(Ryu-Takayanagi)

$\longrightarrow$ extremal surface

HRT entropy, (Hubeny-Rangamani-Takayanagi)

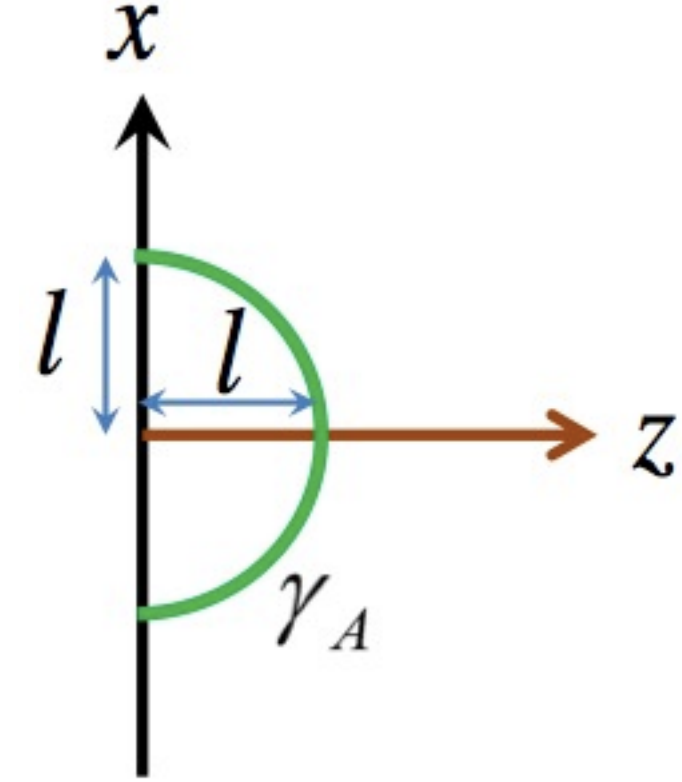
Hubeny-Rangamani-Takayanagi, 07

HEE from AdS3/CFT2

AdS3 metric:

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + dz^2 + dx^2), \qquad \text{where } z = 1/r.$$

Consider constant time slice, the minimal area "surface" is given by the half-circle on (x,z) plane.



$$L(\gamma_A) = 2R\int_a^l dz \frac{l}{z\sqrt{l^2 - z^2}} = 2R\log\frac{2l}{a}.$$

$\longrightarrow$ HEE $S_A = \dfrac{L(\gamma_A)}{4G_N^{(3)}} = \dfrac{c}{3}\log\dfrac{2l}{a}, \quad \text{where } c \equiv \dfrac{3R}{2G_N^{(3)}}.$

agree with QFT.

HEE from AdS3/CFT2

At finite temperature T:

Suppose high T, $l/\beta \gg 1$, where $\beta = 1/T$.

One has to consider BTZ blackhole background, with temperature T.

metric:

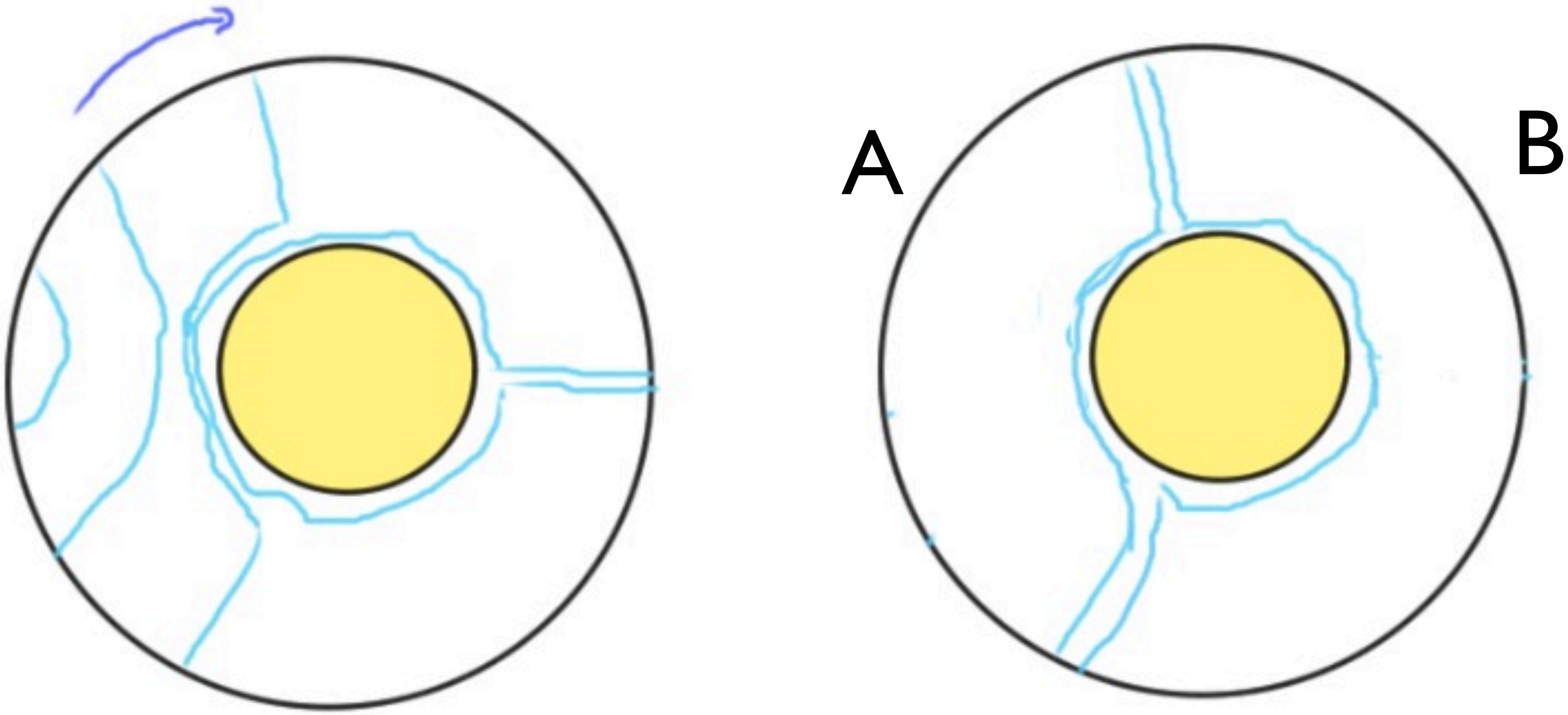
$$ds^2 = -(r^2 - r_H^2)dt^2 + \frac{R^2}{r^2 - r_H^2}dr^2 + r^2 d\phi^2,$$

Similarly, calculate HEE

$$S_A = \frac{c}{3}\log\left(\frac{\beta}{a}\sinh\left(\frac{\pi l}{\beta}\right)\right).$$

Agree with QFT!

HEE for black holes



- small interval, feels only asymptotic AdS
  large interval, more contributions from thermal entropy/black hole
- Connections with Bekenstein-Hawking entropy.
- Can have multiple surfaces ending on the same boundary.

  $S_A \neq S_B$, a mixed state instead of a pure one.

Properties of RT entropy

- Existence of minimal area surface
  Uniqueness
  Continuity of S_A
- $S_A>0$
- Subadditivity $S_A+S_B \geq S_{AB}$
- Araki-Lieb/triangle inequality: $|S_A-S_B| \leq S_{AB}$
- Strong subadditivity(SSA):
  $S_{AB}+S_{BC} \geq S_{ABC}+S_B$, $S_{AB}+S_{BC} \geq S_A+S_C$
- Monogamy of mutual information(MMI):

Properties of RT entropy: MMI

MMI:

$$S_{AB} + S_{BC} + S_{AC} \geq S_A + S_B + S_C + S_{ABC}$$

or in mutual informations:

$$I(A:BC) \geq I(A:B) + I(A:C)$$

However, MMI is violated in some QM systems/QFTs,

e.g. 3 qubits: $\rho_{ABC} = \frac{1}{2}(|000\rangle\langle 000| + |111\rangle\langle 111|)$, diagonal

Possible reasons?

(1) probably HEE is wrong

(2) HEE → QM, while tracing over a subsystem, turns QM→classical

MMI: tracing $\Longleftrightarrow$ QM to classical state?

- pure state $S_{total} = 0, \quad \text{while} \quad S_A = S_B \neq 0.$
- Consider another 3-qubit state,

$$\rho'_{ABC} = \frac{1}{2}\big(|000\rangle + |111\rangle\big)\big(\langle 000| + \langle 111|\big),$$

$$= \frac{1}{2}\big(\underbrace{|000\rangle\langle 000| + |111\rangle\langle 111|}_{\rho_{ABC},\ \text{diagonal}} + \underbrace{|000\rangle\langle 111| + |111\rangle\langle 000|}_{\text{off-diagonal}}\big).$$

Tracing over C,

$$\rho_{AB} = \mathrm{Tr}_C\big(\rho'_{ABC}\big) = \frac{1}{2}\big(|00\rangle\langle 00| + |11\rangle\langle 11|\big) = \mathrm{Tr}_C\big(\rho_{ABC}\big),$$

diagonal, thus classically correlated state.

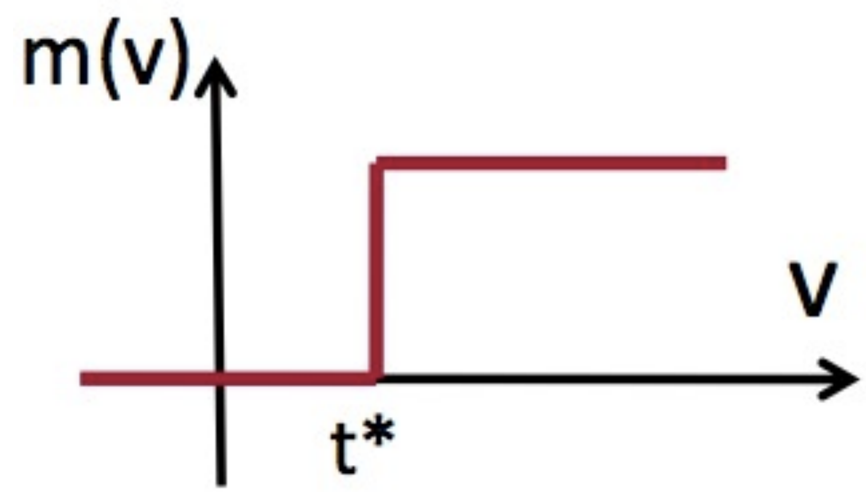
Off-diagonal information lost.

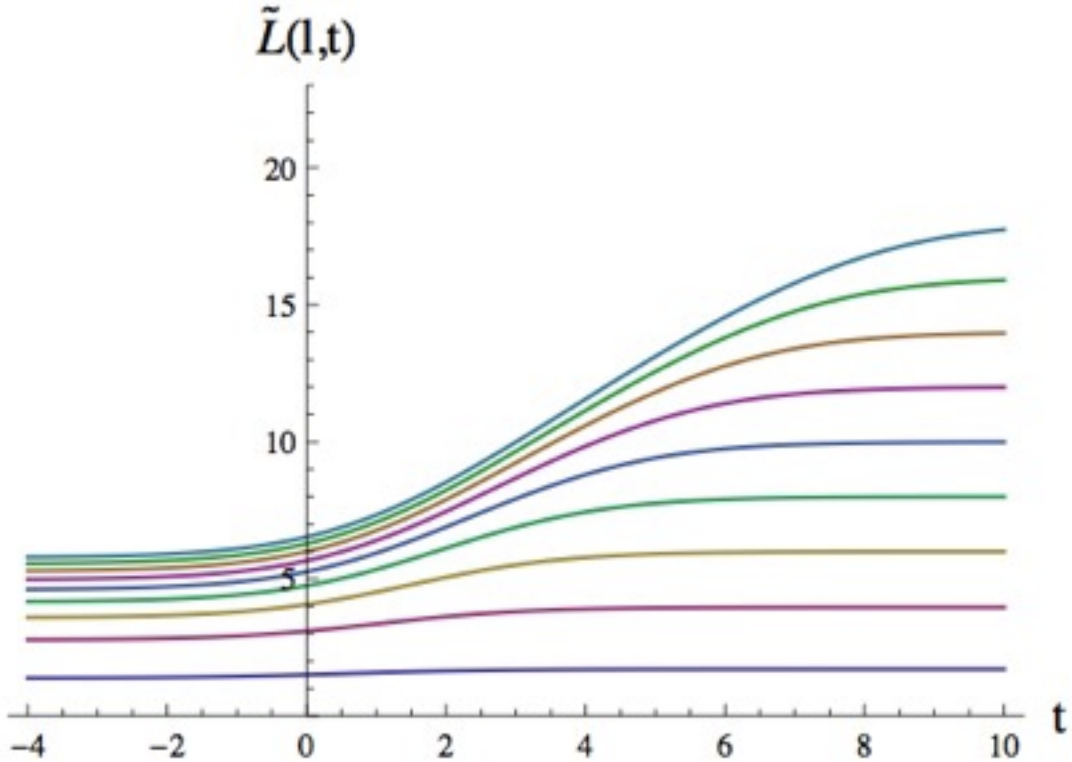
HEE: QM quench & thermalization

One can use thermalization of AdS black hole to mimic QM quench,

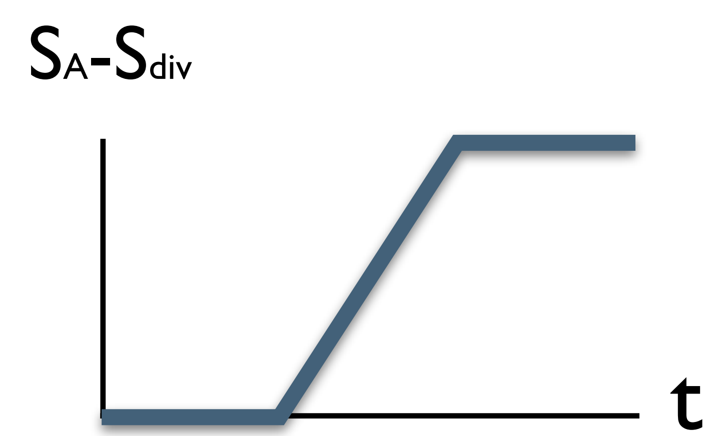
Introduce AdS3-Vaidya metric: Allais-Tonni, 11

$$ds^2 = -(r^2 - m(v))dv^2 + 2drdv + r^2 dx^2,$$





Compare to QFT



RT v.s. HRT formula

RT: static background, constant time slice

Given region A on AdS boundary, to find the minimal area surface

However generally, we would like to write the theory in a covariant version. $\longrightarrow$ HRT

With t involved, not able to find a minimal area surface...

$\longrightarrow$ extremal surface

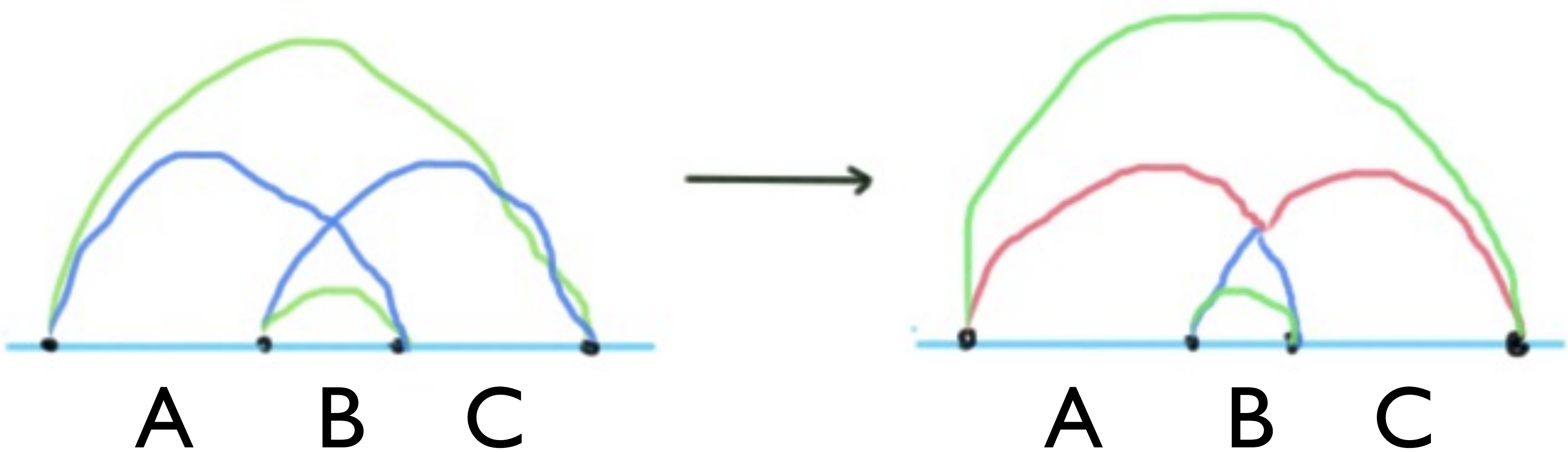
- RT formula is much easier to work with
- Many evidences support RT, e.g. Lewkowycz-Maldacena, 13
- More properties of RT have been proved, while we are not sure whether HRT satisfies these properties.
- Not obvious that RT and HRT always agree with each other, thus either of them might be wrong, needs to be amended, or rigorously discussions is required.

Proof of SSA(RT)

RT: static background, constant time slice

Headrick-Takayanagi, 07

- $S_{AB}+S_{BC} \geq S_{ABC}+S_B$,



From the uniqueness of minimal area surface,

$S(\text{red curve}) \geq S_{ABC}$, $S(\text{blue curve}) \geq S_B$

- $S_{AB}+S_{BC} \geq S_A+S_C$

Similarly for higher dimensions. $\longrightarrow$ Proof of SSA(HRT)?

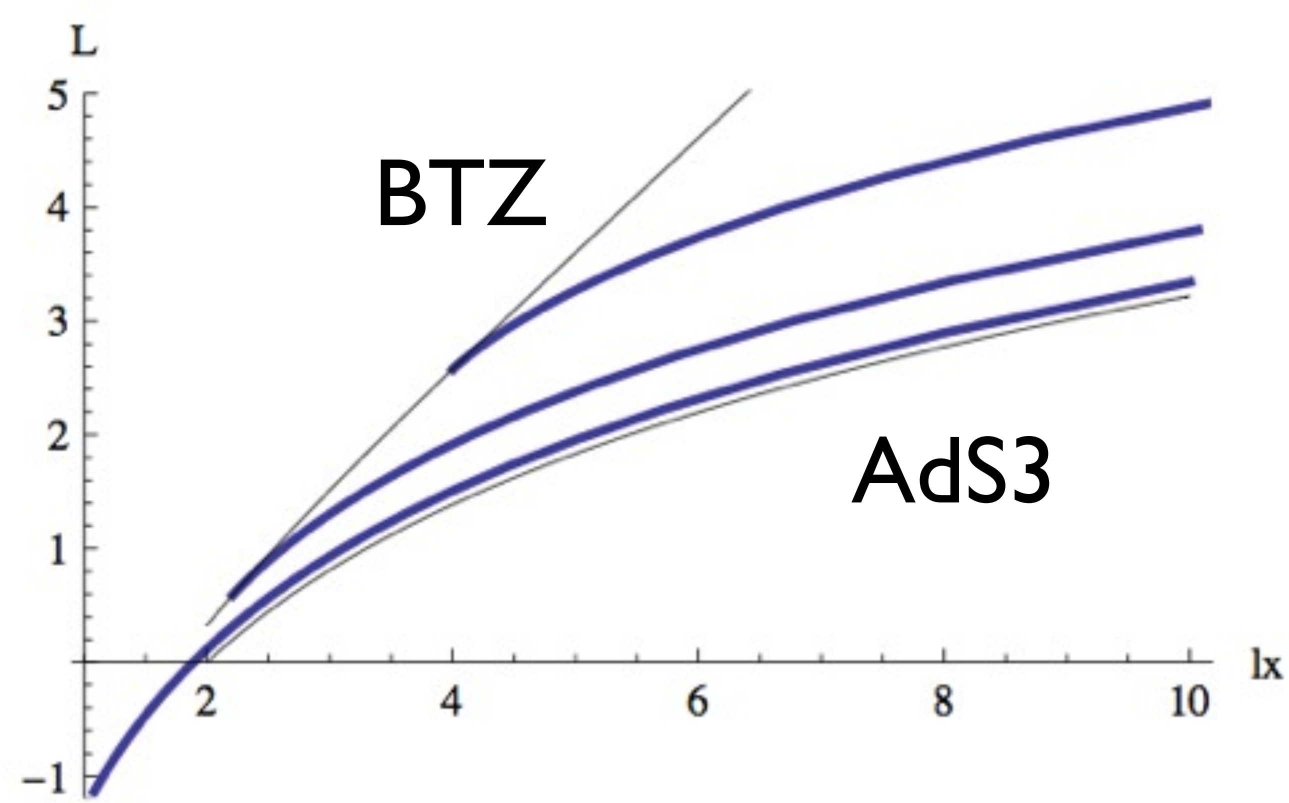
HRT: SSA?

Covariant: (1) time dependent background

(2) not on constant time slice

Working in the bulk, AdS3-Vaidya

A lightlike pulse emitted from boundary at t=0, falls to the center, forming a BTZ black hole. Solve analytically the spacelike geodesic equations.



(1) constant time

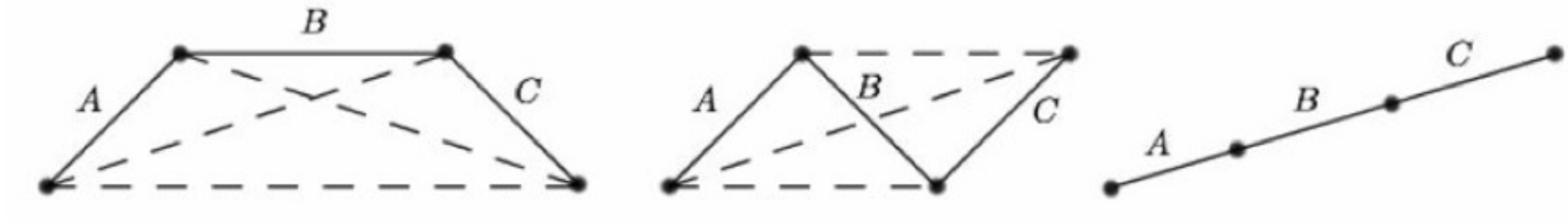
- EE between S_{BTZ} and SAdS
- monotonic, concave

$\longrightarrow$ SSA

Callan-He-Headrick, 12

HRT: SSA? in AdS-Vaidya

General spacelike intervals:



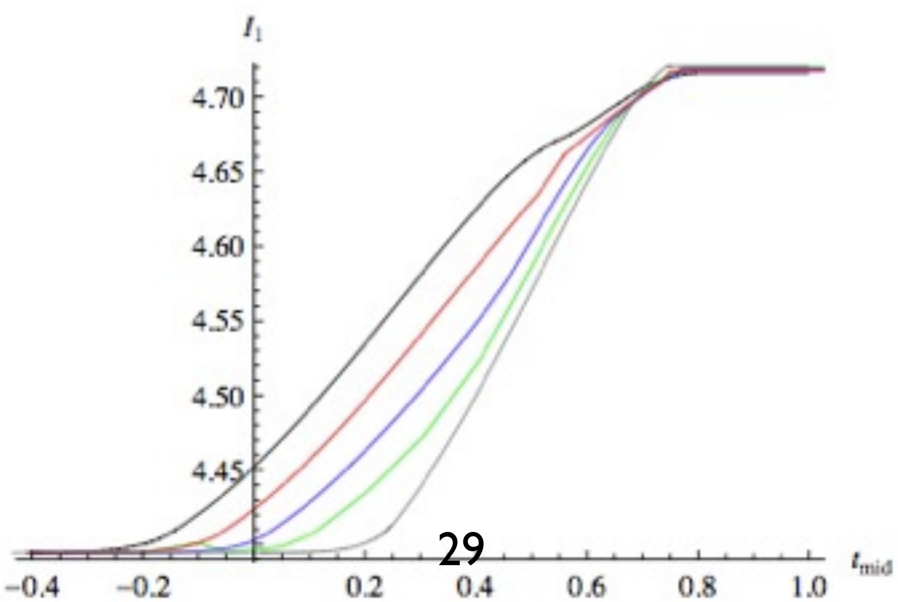
From SSA, introduce

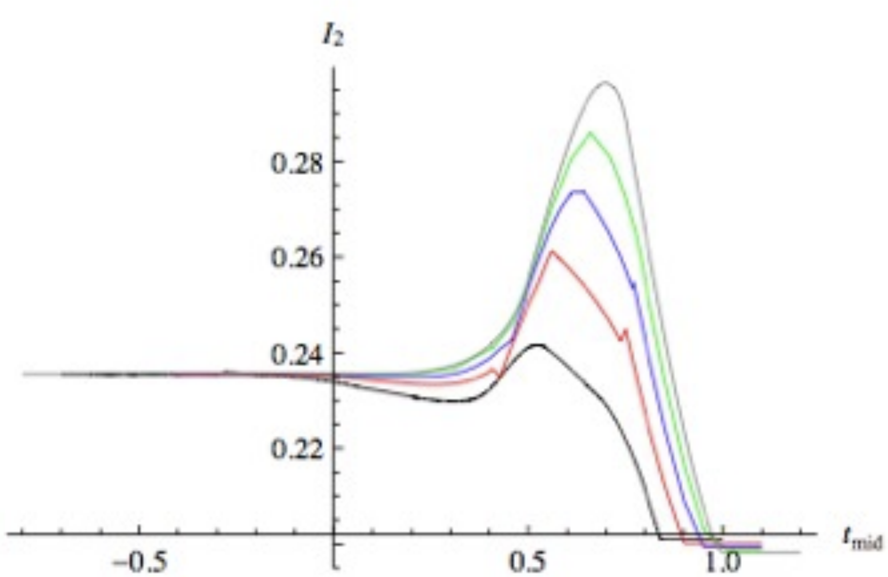
$$I_1(A,B,C) \equiv S_{AB} + S_{BC} - S_A - S_C,$$
$$I_2(A,B,C) \equiv S_{AB} + S_{BC} - S_{ABC} - S_C.$$

SSA requires both $I_{1,2} \geq 0.$

Numerically check examples of the above three combinations, and all have $I_{1,2} \geq 0$

Typical function behavior:





SSA $\Longleftrightarrow$ Null energy condition?

For a general Vaidya metric,

$$ds^2 = -f(r,v)dv^2 + 2drdv + r^2dx^2, \quad f(r,v) = (r^2 - m(v))$$

NEC requires $T_{\mu\nu}n^\mu n^\nu \geq 0,$

where n^μ is lightlike, i.e. $n^\mu n_\mu = 0.$

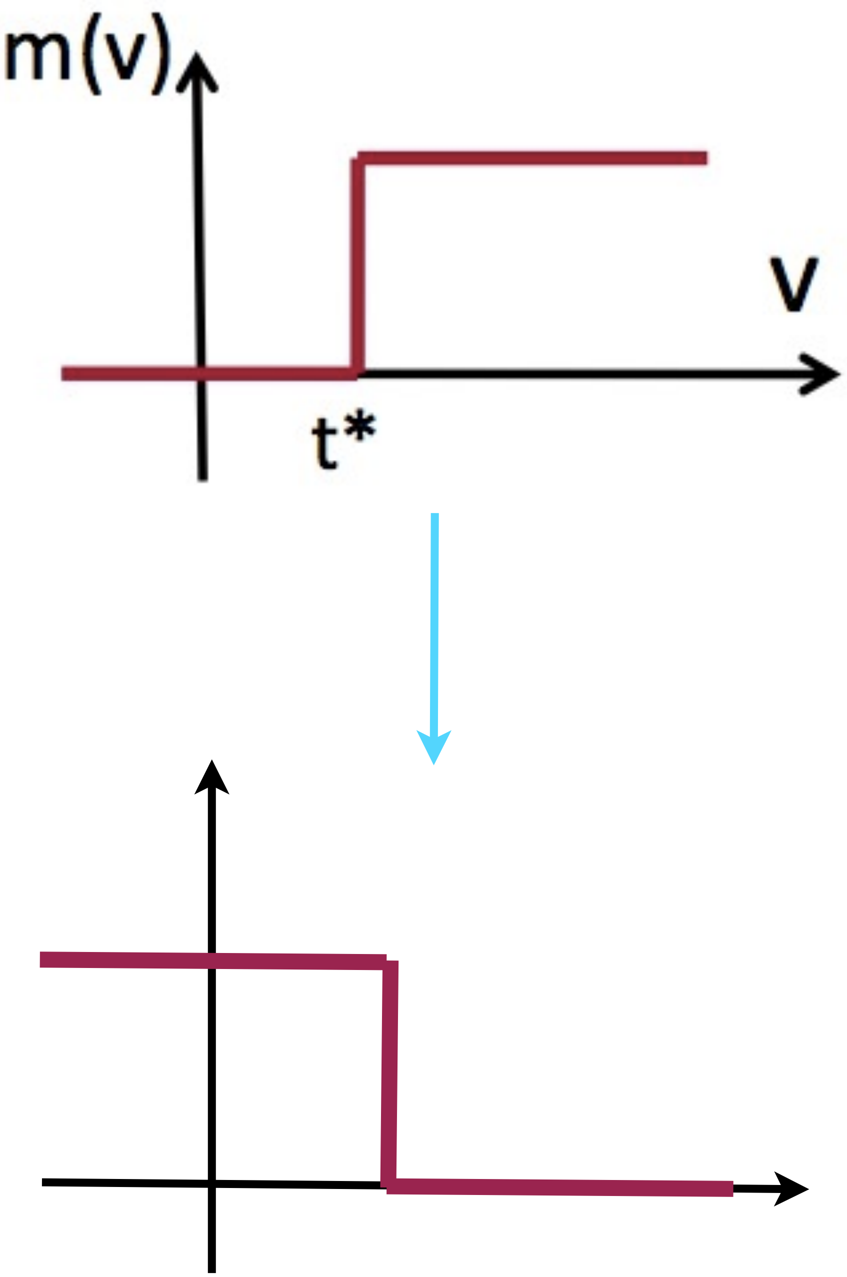
Suppose a null vector $n^\mu = (a, b, 0),$

$\Longrightarrow$ a=0, or b=f(r,v)a/2

The 2nd solution $\Longrightarrow$ $n^\mu \sim (1, f/2, 0)$

$$\Longrightarrow \quad T_{\mu\nu}n^\mu n^\nu = G_{\mu\nu}n^\mu n^\nu = \frac{m'(v)}{2r}$$

Thus NEC $\Longleftrightarrow$ $m'(v) \geq 0$

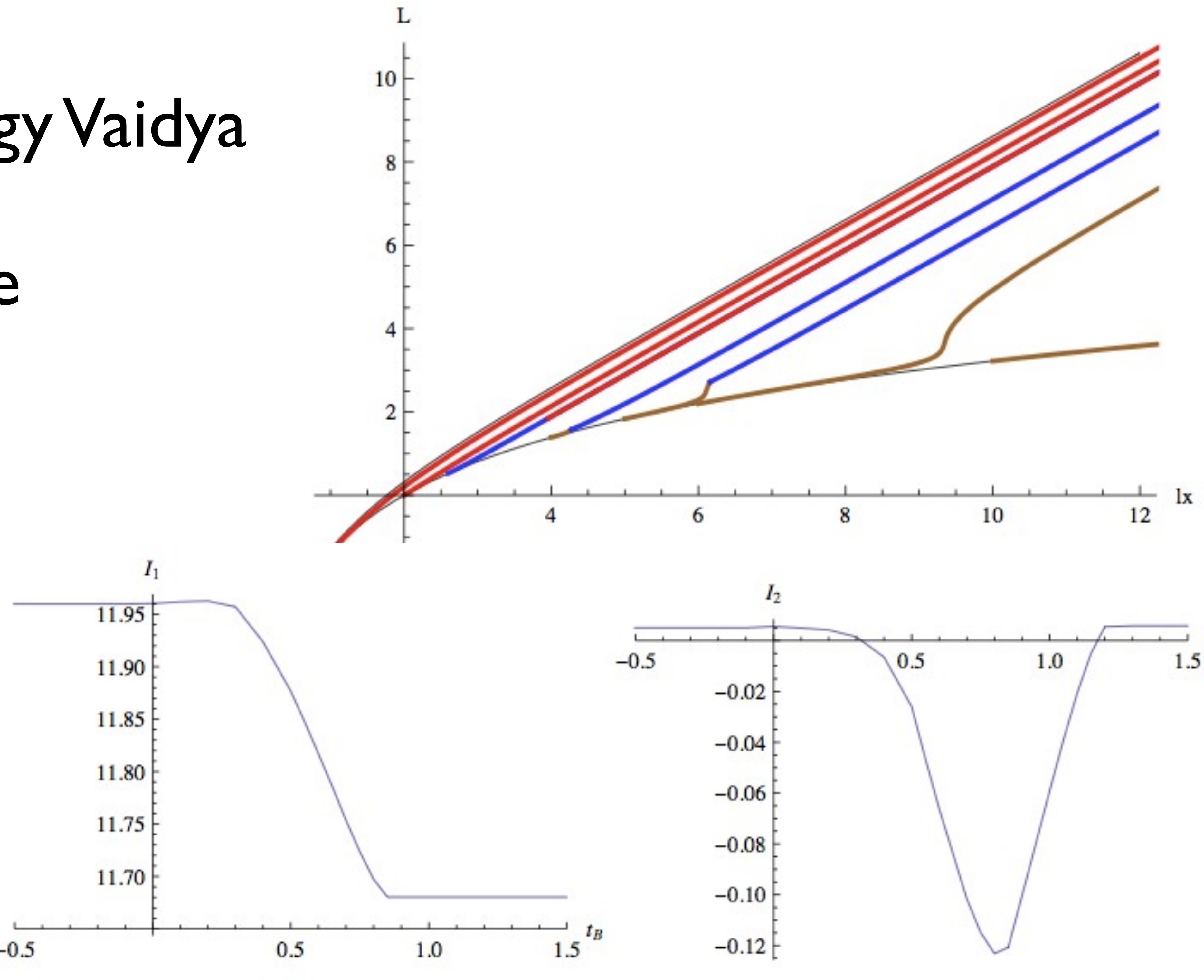


negative-energy Vaidya

SSA $\Longleftrightarrow$ Null energy condition?

negative-energy Vaidya

non-concave



$I_2 < 0$ for some value of t on boundary. SSA violated!

HRT: SSA?

Wall 12

With assumptions of classical, horizonless, globally hyperbolic manifold, and obeying null energy conditon,

Claim: maximin surfaces M(A) = extremal surfaces m(A)

maximin surfaces M(A): First minimizing the area on some achronal slice Σ, and then maximizing the area w.r.t. varying Σ.

Proved SSA and monogamy of mutual information in this case, with more nice hypersurface assumptions.

- too many assumptions, not very sure about its generic validity
- bulk with horizon?

Various hypersurfaces

- $\mathscr{W}$, extremal surfaces HRT, 07

- $\mathscr{Y}$, surface with vanishing null expansion along future and past light-sheets

- $\mathscr{X}$, minimal-area surface on maximal-area slice of the bulk, or "minimax" surface

- $\mathscr{Z}$, causal construction

It's argued $\mathscr{W}=\mathscr{Y}$, while $\mathscr{X}$ could be consistent with the two in some specific cases, for example static metric. $\mathscr{Z}$ generally does not coincide with $\mathscr{W}$, but provides a easily calculated bound.

Note: $\mathscr{X}$ is similar to "maximin" surface, which indicates a requirement of rigorous check of previous proof.

Penetration through the horizon?

Universe Traveler Rules #1:

There is NO turning back through horizons!

Classical gravity: One can never access the information behind horizons.

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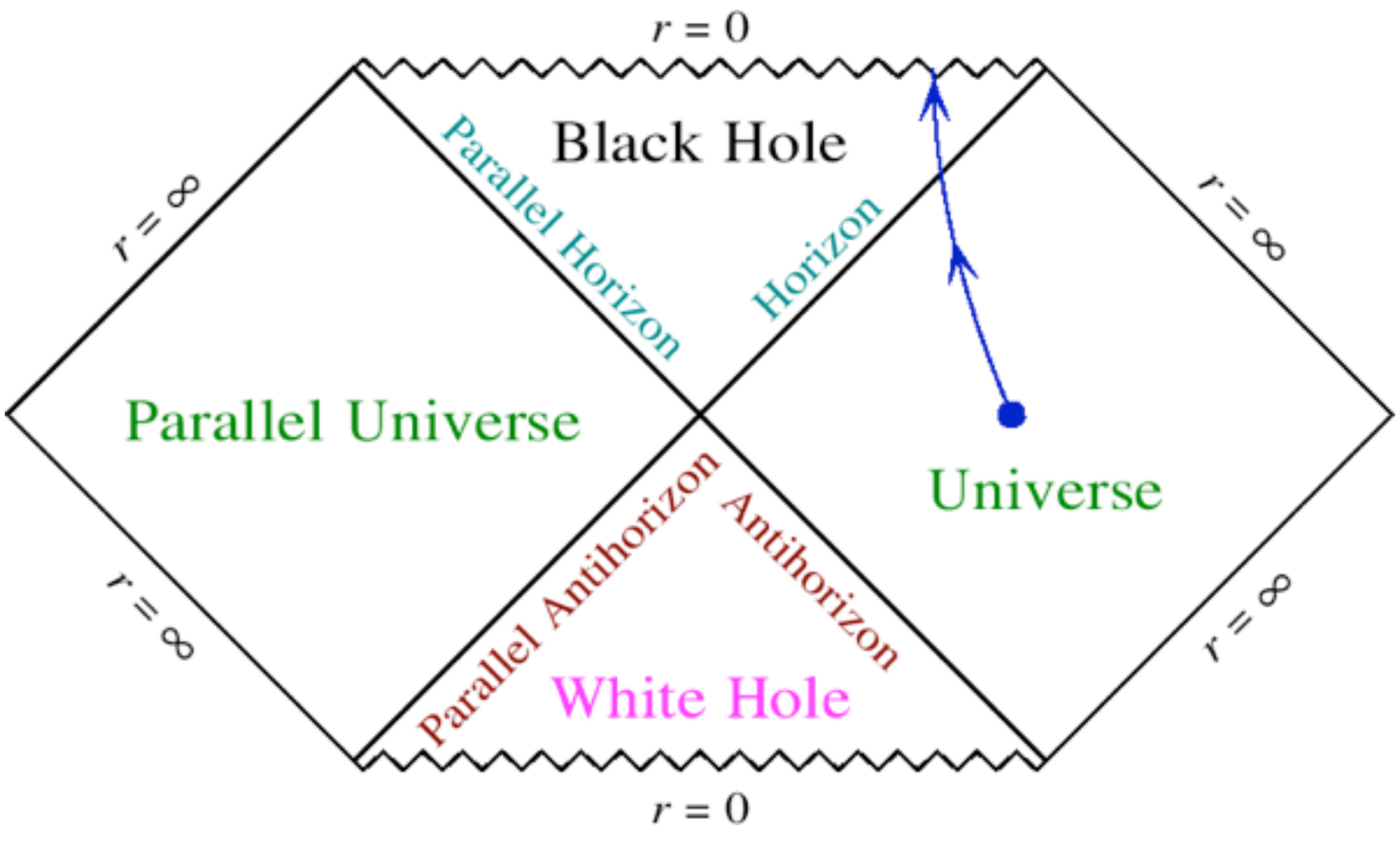

timelike ...

Oh, NO!!!

Penetration through the horizon?

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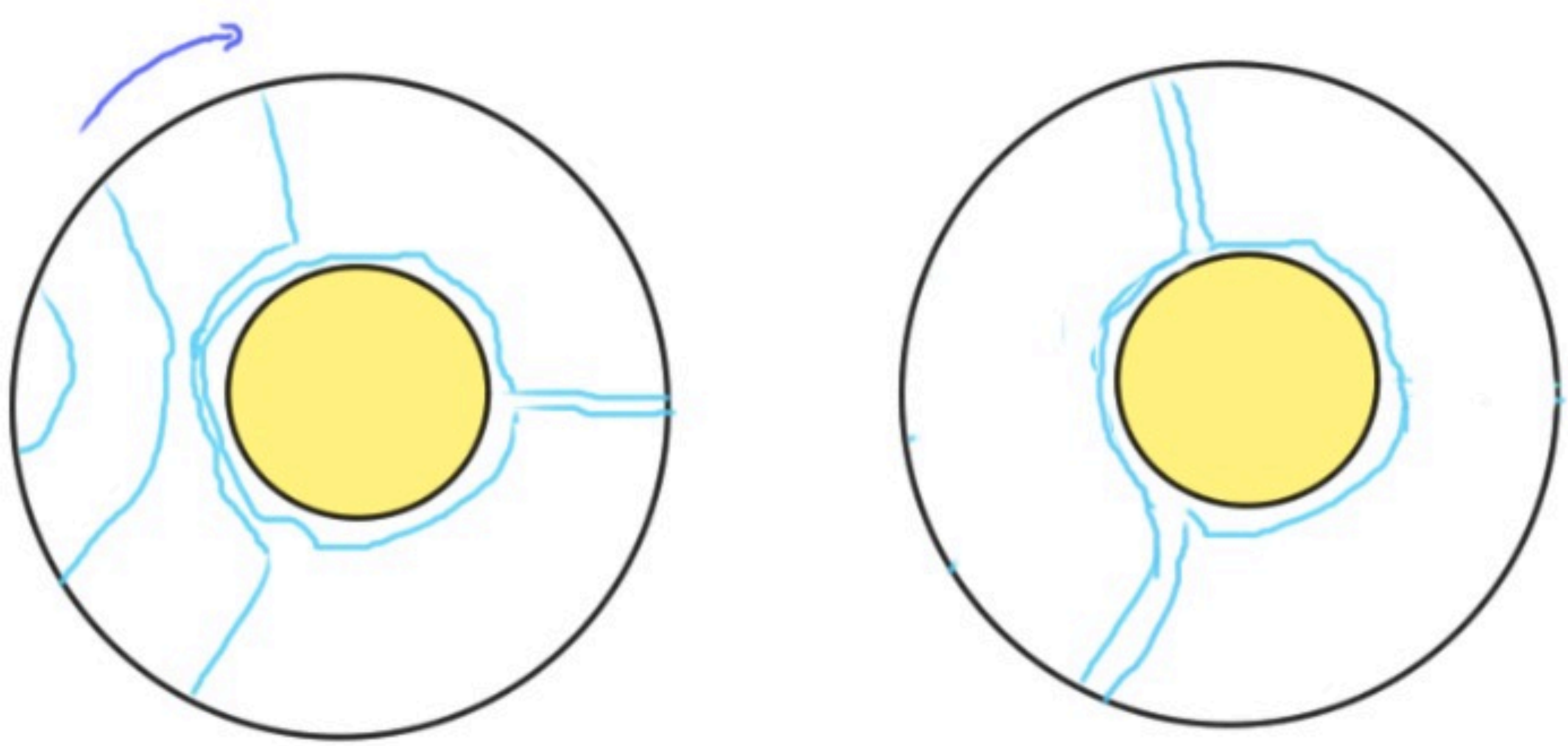
Classical gravity: One can never access the information behind horizons.



spacelike geodesic wrapping around the horizon.

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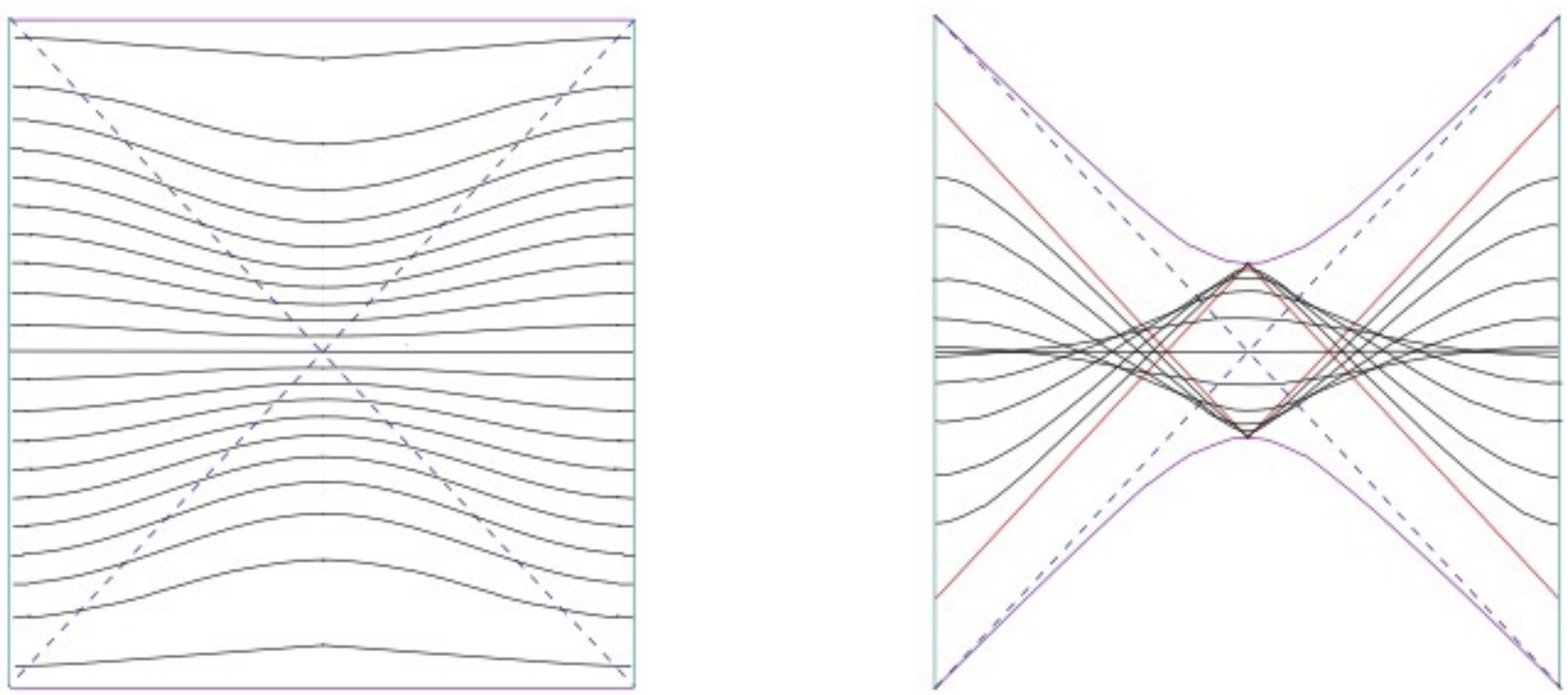


spacelike geodesics connecting the two copies of boundaries together, through the horizon.

Penetration through the horizon?

Universe Traveler Rules #1:

There is NO turning back through horizons!

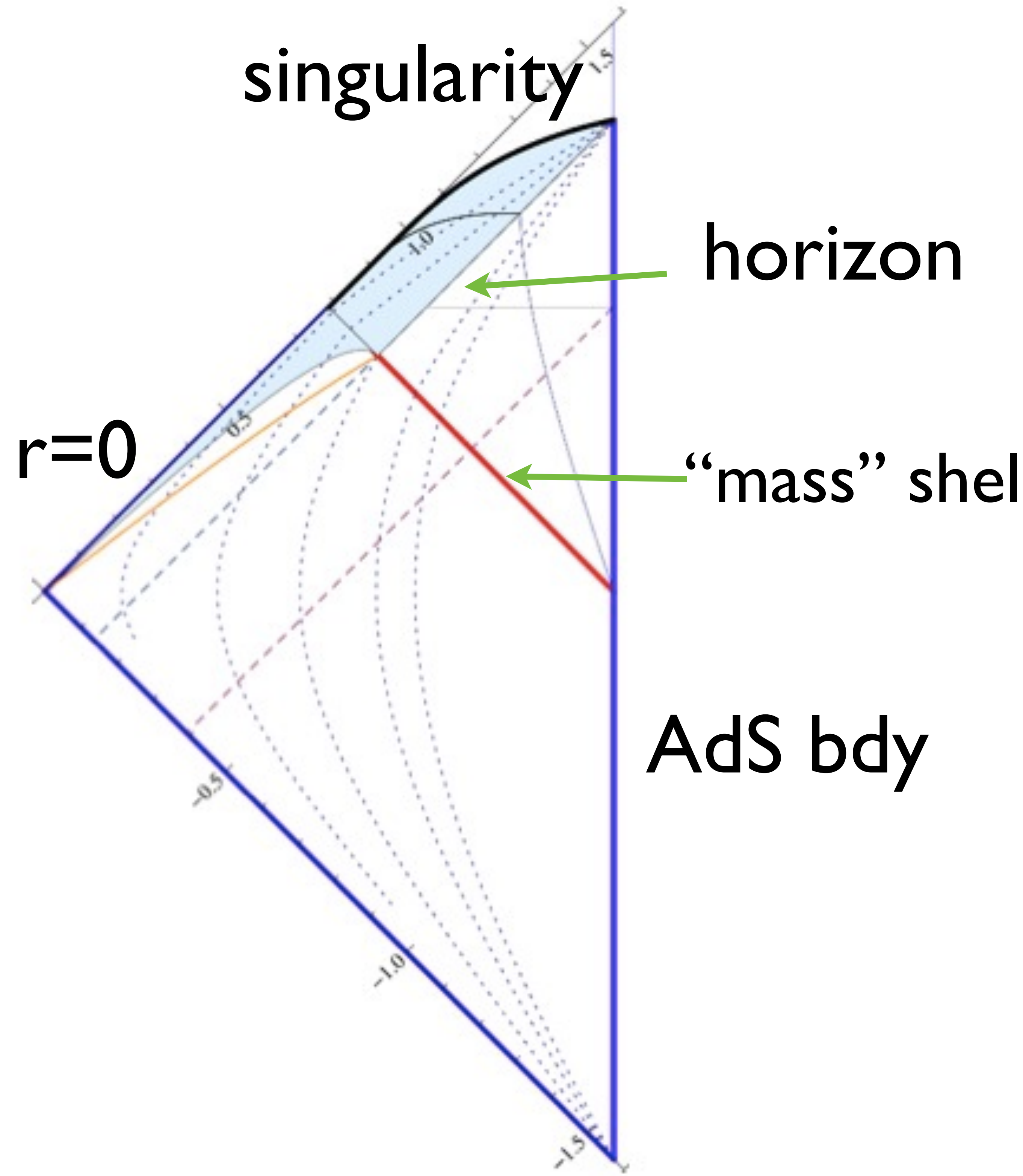
Classical gravity: One can never access the information behind horizons.

Many works support the argument, e.g. Hubeny 12:

"In a static black hole spacetime, no extremal surface (of any dimensionality, anchored on any region in the boundary) can ever penetrate the horizon."

Penetration through the horizon?

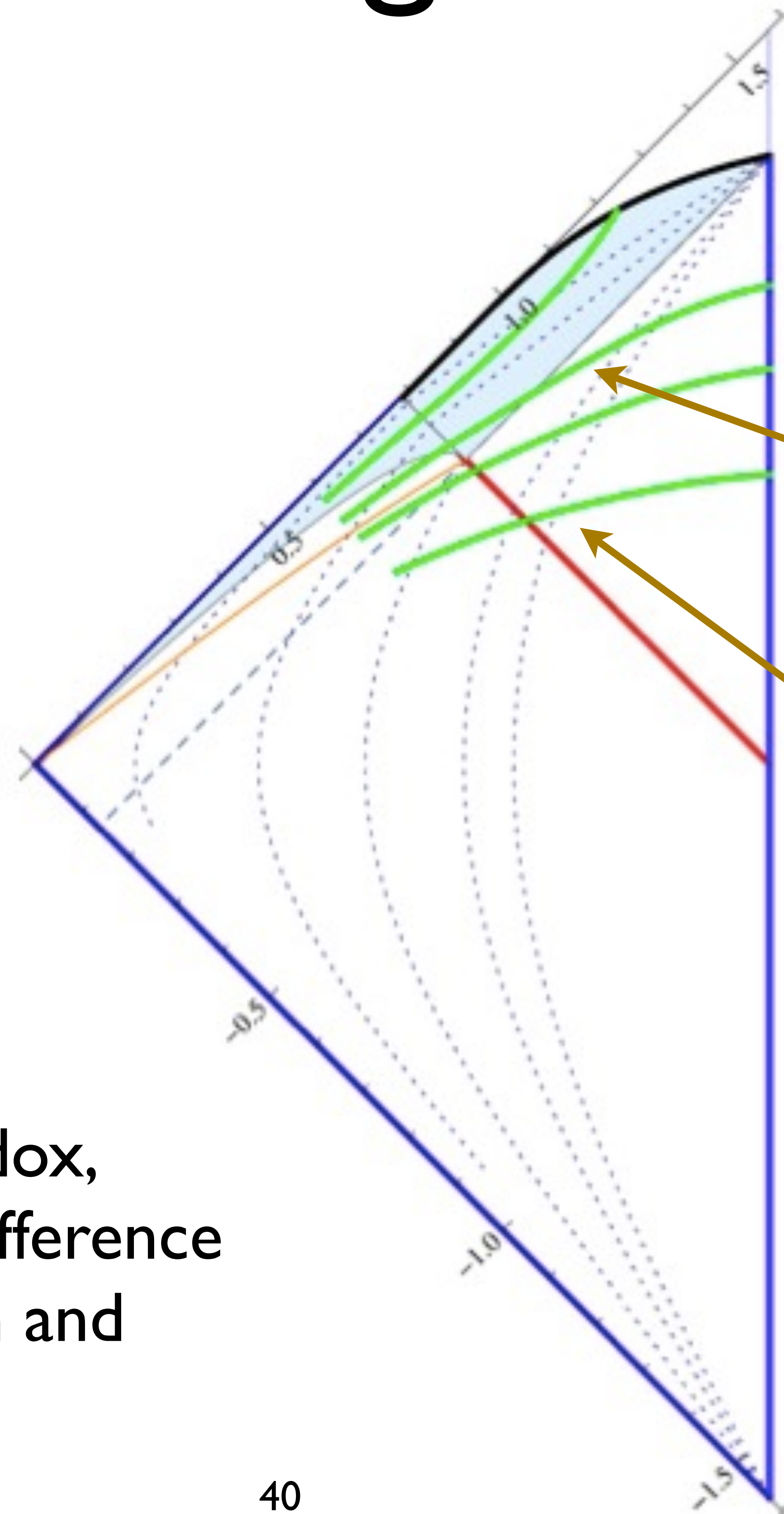
AdS3-Vaidya space



Penetration through the horizon?

AdS3-Vaidya space

spacelike geodesics anchoring on AdS bdy at positions with the same time



Might be useful to

- blackhole's information paradox,
- causality v.s. HEE, note the difference between causal construction and extremal surface

Summary

- AdS/CFT correspondence: very brief introduction
- EE in QFT: EE, Renyi entropy, properties(SSA, area law), replica method, QM quench
- HEE: RT, HRT, properties, AdS3/CFT2, time evolution,
- SSA: proof of RT, HRT(?), SSA vs NEC
- penetration through horizons

Interesting Projects

- Since RT was "proved" from AdS/CFT, how about HRT?
- covariant SSA?
- RT and/or HRT correct?
- HEE's application on holographic RG flow
- HEE in holographic CMT

Thank You!