On short interval expansion of Rényi entropy

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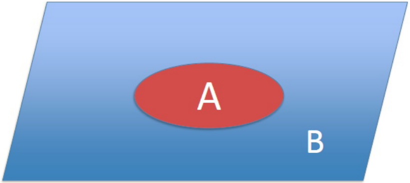
Based on the work with Jia-Ju Zhang, 1309.5453

IHEP, Beijing, Oct. 10, 2013

Entanglement entropy



- Divide the system to be A and B such that $\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B$
- Reduced density matrix: $\rho_A = \mathrm{tr}_B \rho_{tot}$
- von Neumann Entanglement entropy: $S_A = -\mathrm{tr}\rho_A \ln \rho_A$
- It is the entropy for an observer who is only accessible to A and not to B
- Properties:
  1. For pure state $S_A = S_B$, otherwise $S_A \neq S_B$
  2. Strong subadditivity: $S_{A+B+C} + S_B \leq S_{A+B} + S_{B+C}$
  3. Subadditivity: $S_{A+B} \leq S_A + S_B$

Physical implication

- It is hard to be observed directly in Lab.
- It has been computed numerically in CM systems: spin chains, lattice models, ...
- Encodes valuable information of the system: dynamical d.o.f.
- Various applications: as quantum order parameter in CM, characterize non-equilibrium states,...
- A bridge between gravity and CM

Rényi entropy

- More generally one can define the Rényi entanglement entropy, or in short the Rényi entropy, of A and B as

$$S_A^{(n)} = -\frac{1}{n-1} \log \mathrm{Tr}_A \rho_A^n. \tag{1.1}$$

- It is easy to see that the entanglement entropy and the Rényi entropy are related by

$$S_A = \lim_{n \to 1} S_A^{(n)}. \tag{1.2}$$

- The relation provides a practical way to compute EE

Rényi mutual information

- Choose two subsystems A and B which are not necessarily each other's complement
- Define the Rényi mutual information of A and B

$$I^{(n)}_{A,B} = S^{(n)}_A + S^{(n)}_B - S^{(n)}_{A\cup B}. \tag{1.3}$$

- Free from UV and IR divergences
- For $n = 1$, it is called mutual information, which measures an entropic correlation between A and B
- From subadditivity, we know $I(A, B) \geq 0$

EE in QFT

- Consider a QFT on a $(d+1)$-dim. manifold $R \times M$, where R is time direction
- Choose subsystem by a d-dim. submanifold $A \in M$ at a fixed time
- In this case, the EE S_A is called the geometric entropy as it depends on the geometry of A L.Bombelli et.al. 1986, M. Srednicki 9304048

$$S_A = \gamma \frac{\text{Area}(\partial A)}{\epsilon^{d-1}} + \text{subleading terms} \tag{1.4}$$

  where ∂A is the boundary of A, ϵ is the UV cutoff and γ is a constant depending on the system
- This suggests that entanglement between A and B occurs at the boundary most strongly
- The Rényi entropy could be defined similarly
- In a sense, the entanglement entropy is a generalization of "Wilson loop"
- It is really hard to compute in QFT, even for free field theory

Exception: 2D CFT

- For A being a single interval of length l C. Holzhey et.al. 9403108

$$S_A = \frac{c}{3} \log \frac{l}{\epsilon} \tag{1.5}$$

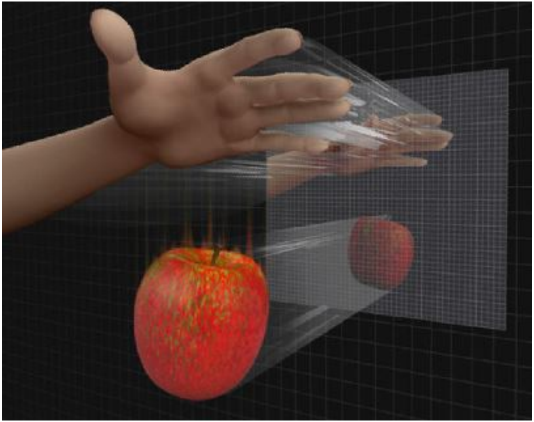
where c is the central charge

- The situations of a compactified circle or an infinite system at finite temperature could be treated by using the conformal map
- Rényi entropy P. Calabrese and J.L. Cardy 0405152

$$S_n = \frac{c}{6}\left(1 + \frac{1}{n}\right) \log \frac{\ell}{\epsilon}, \tag{1.6}$$

AdS/CFT correspondence

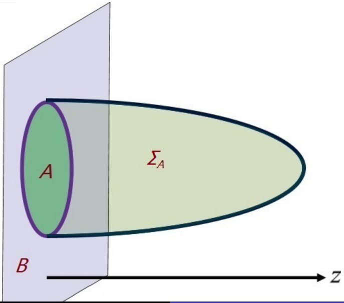
- Quantum gravity in AdS spacetime is dual to a CFT at AdS boundaryJ. Maldacena 1997
- A concrete realization of holographic principle



Holographic entanglement entropy Ryu and Takayanagi 2006

- AdS/CFT: A field theory could be holographically described by a higher-dim. gravity
- Ryu and Takayanagi(2006): Find a codimension two minimal surface Σ_A in the bulk that is homogeneous to A
- The entanglement entropy (for Einstein gravity)

$$S_A = \frac{\mathrm{Area}(\Sigma_A)}{4G_N}$$

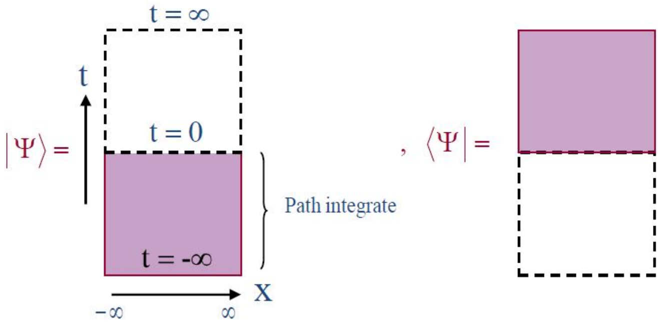


Remarks on HEE

- RT formula has passed some nontrivial tests
  1. Satisfies strong subadditivity
  2. Reproduce one interval EE in 2D CFT
- It has been intensely studied since its proposal
- In higher dimension ($d \geq 3$), it has been shown recently by A. Lewkowyca and J. Maldacena (1304.4926) (see also D.V. Fursaev (0606184) and H. Casini et.al. (1102.0440)) but the proof has not been well-accepted
- In $2+1$ dimension, RT formula has been proven recently by T. Hartman (1303.6955) and T. Faulkner (1303.7221) independently

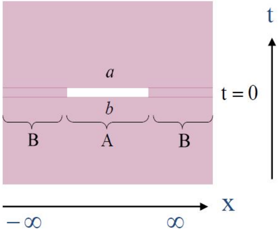
Replica trick

- The standard way is to use replica trick J. Callan et.al. 9401072
- Here, we only focus on the 2D CFT, which provides more analytic results
- In Euclidean path-integral, the ground state wave-functional is represented by T. Takayanagi's lecture in 7th Asian winter school

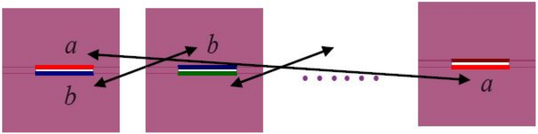


Replica trick II

$[\rho_A]_{ab} =$



$\mathrm{Tr}\,(\rho_A)^n =$



= a path integral over

n-sheeted Riemann surface Σ_n

Replica trick III

- Replica trick: computation in product orbifold $(\mathrm{CFT})_n/Z_n$
- Branch points: twist operators with dimension

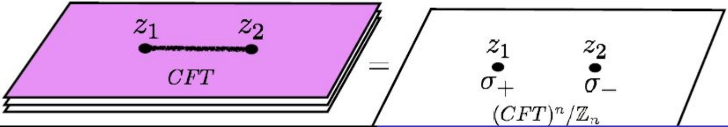
$$h=\bar{h}=\frac{c}{24}\left(n-\frac{1}{n}\right). \tag{2.1}$$

- One interval case

$$\mathrm{Tr}\rho_A^n=\langle\sigma(\ell,\ell)\tilde{\sigma}(0,0)\rangle_C=c_n\ell^{-\frac{c}{6}\left(n-\frac{1}{n}\right)}, \tag{2.2}$$

from which the Rényi entropy for one interval could be read P. Calabrese and J.L. Cardy 0405152

$$S_n=\frac{c}{6}\left(1+\frac{1}{n}\right)\log\frac{\ell}{\epsilon}, \tag{2.3}$$

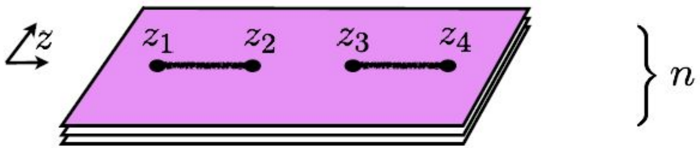


Multi-intervals

- In the case of N intervals, there are more branch cuts so that the Riemann surface is of genus $(n-1)(N-1)$, where n is the number of replica
- If we have multiple intervals $A = [z_1, z_2] \cup \cdots \cup [z_{2N-1}, z_{2N}]$,

$$\mathrm{Tr}\rho_A^n = \langle \sigma(z_{2N}, \bar{z}_{2N})\tilde{\sigma}(z_{2N-1}, \bar{z}_{2N-1}) \cdots \sigma(z_2, \bar{z}_2)\tilde{\sigma}(z_1, \bar{z}_1)\rangle_C.$$

- It is very difficult to compute
- Nevertheless, in the case that the intervals are short, we may use operator product expansion(OPE) to compute



Proof of RT formula in AdS_3: A sketch T. Faulkner 1303.7221

- Find the bulk gravity solutions B^γ such that $\partial B^\gamma = \Sigma_n$
- Key point: all solutions of AdS_3 gravity

$$B^\gamma = H_3/\Gamma_\gamma$$

  where Γ_γ is the subgroup of isometry $SL(2, C)$
- In the classical gravity limit, keep only the solution of least action
- Consider the handlebody solutions, preserving the boundary replica symmetry
- This requires that Γ_γ is the schottky group
- Monodromies of the cycel gives the quotient
- The conformal Ward identity gives the bulk action
- An independent proof by T. Hartman (1303.6955) used the CFT techniques

Quantum correction

- For large separation, the mutual information is vanishing

- The mutual information satisfies M. Wolf et.al. 0704.3906

$$I(A,B) \geq \frac{|<\mathcal{O}_A \cdot \mathcal{O}_B> - <\mathcal{O}_A><\mathcal{O}_B>|^2}{2|\mathcal{O}_A|^2|\mathcal{O}_B|^2} \tag{2.4}$$

- $I(A,B)$ is only vanishing to the leading order in G_N
- It should be nonzero, with quantum corrections T. Faulkner et.al. 1307.2892
- With the bulk solution, the 1-loop quantum correction to Rényi entropy has been computed T. Barrella et.al. 1306.4682

Question

- Can one find the 1-loop quantum correction from CFT side?
- In principle, this is feasible
- Recall that in AdS_3/CFT_2, $c = \frac{3l}{2G}$
- In the large c limit, we may recover the weak gravity result, even with quantum correction
- In practice, as the EE is nonlocal, we only manage to compute the Rényi entropies in the small interval limit, which allows us to use OPE techniques
- The results are really remarkable

Correlators in 2D CFT

- In a 2D CFT, all the operators could be written in terms of quasiprimary fields and their derivatives
- We write the quasiprimary operators as ϕ_i with conformal weights h_i and $\bar{h}_i$
- The correlation functions of two and three quasiprimary operators on complex plane C are

$$\langle \phi_i(z_i,\bar{z}_i)\phi_j(z_j,\bar{z}_j)\rangle_C = \frac{\alpha_i\delta_{ij}}{z_{ij}^{2h_i}\bar{z}_{ij}^{2\bar{h}_i}},$$

$$\langle \phi_i(z_i,\bar{z}_i)\phi_j(z_j,\bar{z}_j)\phi_k(z_k,\bar{z}_k\rangle_C$$

$$= \frac{C_{ijk}}{z_{ij}^{h_i+h_j-h_k} z_{jk}^{h_j+h_k-h_i} z_{ik}^{h_i+h_k-h_j} \bar{z}_{ij}^{\bar{h}_i+\bar{h}_j-\bar{h}_k} \bar{z}_{jk}^{\bar{h}_j+\bar{h}_k-\bar{h}_i} \bar{z}_{ik}^{\bar{h}_i+\bar{h}_k-\bar{h}_j}},$$

with $z_{ij} \equiv z_i - z_j$ and $\bar{z}_{ij} \equiv \bar{z}_i - \bar{z}_j$.

OPE in 2D CFT

The OPE of two quasiprimary operators could be generally written as

$$\phi_i(z,\bar{z})\phi_j(0,0) = \sum_k C_{ij}^k \sum_{m,r\geq 0} \frac{a_{ijk}^m}{m!}\frac{\bar{a}_{ijk}^r}{r!}\frac{1}{z^{h_i+h_j-h_k-m}\bar{z}^{\bar{h}_i+\bar{h}_j-\bar{h}_k-r}}\partial^m\bar{\partial}^r\phi_k(0,$$

where the summation k is over all quasiprimary operators and there are definitions

$$a_{ijk}^m \equiv \frac{C_{h_k+h_i-h_j+m-1}^m}{C_{2h_k+m-1}^m}, \quad \bar{a}_{ijk}^r \equiv \frac{C_{\bar{h}_k+\bar{h}_i-\bar{h}_j+r-1}^r}{C_{2\bar{h}_k+r-1}^r}, \quad C_{ij}^k \equiv \frac{C_{ijk}}{\alpha_k}$$

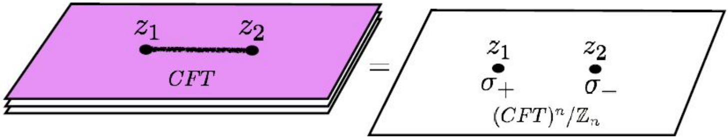
with the binomial coefficient being $C_x^y = \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}$.

CFT$_n$

- The replica trick requires us to study a orbifold CFT: $(\text{CFT})_n/Z_n$
- The CFT_n has central charge nc with c being the central charge of CFT_1, and the stress tensors are

$$\sum_{j=0}^{n-1} T(z_j), \quad \sum_{j=0}^{n-1} \bar{T}(\bar{z}_j) \tag{3.1}$$

where $T(z_j)$, $\bar{T}(\bar{z}_j)$ are the stress tensors of the j-th copy the original CFT and z_j is the coordinate of the j-th copy of the Riemann surface $\mathcal{R}_{n,N}$.



Quasiprimaries in CFT$_n$

We denote the linear independent quasiprimary operators of CFT_n as $\Phi_K(z,\bar{z})$ with conformal wights h_K and $\bar{h}_K$. The product of quasiprimary operators in each copy forms a quasiprimary operator of CFT_n,

$$\Phi_K(z,\bar{z}) = \prod_{j=0}^{n-1} \phi_{k_j}(z_j,\bar{z}_j), \tag{3.2}$$

and in this case there are

$$K = \{k_j\}, \quad \alpha_K = \prod_{j=0}^{n-1} \alpha_{k_j}, \quad h_K = \sum_{j=0}^{n-1} h_{k_j}, \quad \bar{h}_K = \sum_{j=0}^{n-1} \bar{h}_{k_j}. \tag{3.3}$$

Note that not all of the quasiprimary operators of CFT_n could be written in the above form.

General prescription M. Headrick 1006.0047, P. Calabrese et.al. 1011.5482, BC and J-j Zhang 1309.5453

When the intervals are short, we have the OPE of the twist operators

$$\sigma(z,\bar{z})\tilde{\sigma}(0,0) = c_n \sum_K d_K \sum_{m,r\geq 0} \frac{a_K^m}{m!} \frac{\bar{a}_K^r}{r!} \frac{1}{z^{2h-h_K-m}\bar{z}^{2\bar{h}-\bar{h}_K-r}} \partial^m \bar{\partial}^r \Phi_K(0,0), \tag{3.4}$$

with the summation K being over all the independent quasiprimary operators of CFT_n. Here

$$a_K^m \equiv \frac{C^m_{h_K+m-1}}{C^m_{2h_K+m-1}}, \quad \bar{a}_K^r \equiv \frac{C^r_{\bar{h}_K+r-1}}{C^r_{2\bar{h}_K+r-1}}. \tag{3.5}$$

- For a quasiprimary operator Φ_K, the OPE coefficient is

$$C_K = c_n \ell^{-\frac{c}{6}\left(n-\frac{1}{n}\right)} d_K, \tag{3.6}$$

- The OPE coefficient of its derivatives $\partial^m \bar{\partial}^r \Phi_K$ is

$$C_K^{(m,r)} = c_n \ell^{-\frac{c}{6}\left(n-\frac{1}{n}\right)+m+r} d_K \frac{a_K^m}{m!} \frac{\bar{a}_K^r}{r!}. \tag{3.7}$$

Remarks

- For a concrete CFT model, the summation should be over all the conformal blocks
- For pure AdS_3 gravity, it is enough to consider the vacuum Verma module
- The OPE of the twist operators could be represented by a diagram

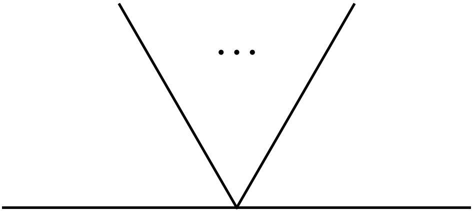


Figure: OPE vertex of twist operators

How to compute the OPE coefficients

- For usual OPE, it depends on the three point functions
- For the OPE of twist operators, we may just focus on the one interval case, in the small interval limit P. Calabrese et.al. 1011.5482
- When there is one interval $A = [0, \ell]$, we consider the expectation value of one quasiprimary operator $\Phi_K(z, \bar{z})$ on $\mathcal{R}_{n,1}$, and then we have

$$\frac{Z_n(A)}{Z^n} \langle \Phi_K(z, \bar{z}) \rangle_{\mathcal{R}_{n,1}} = \langle \Phi_K(z, \bar{z}) \sigma(\ell, \ell) \tilde{\sigma}(0, 0) \rangle_C. \quad (3.8)$$

- Using the OPE of twist operators and the orthogonality of quasiprimary operators of CFT_n we have

$$d_K = \frac{1}{\alpha_K \ell^{h_K + \bar{h}_K}} \lim_{z \to \infty} z^{2h_K} \bar{z}^{2\bar{h}_K} \langle \Phi_K(z, \bar{z}) \rangle_{\mathcal{R}_{n,1}}, \quad (3.9)$$

with α_K being a normalization coefficient.

- The key ingredients in the OPE of twist operators is to calculate the coefficients α_K and d_K.

Holomorphic quasiprimary operators in CFT_1

Explicitly the holomorphic quasiprimary operators of first few levels are listed as follows.

- At level 0, it is the identity operator 1.
- At level 2, there is one quasiprimary operator the stress tensor T.
- At level 4, it is $\mathcal{O} = (TT) - \frac{3}{10}\partial^2 T$.
- At level 6, they are $\mathcal{Q} = (\partial T \partial T) - \frac{2}{9}\partial^2(TT) + \frac{1}{42}\partial^4 T$ and $\mathcal{R} = \mathcal{P} + \frac{9(14c+43)}{2(70c+29)}\mathcal{Q}$, with $\mathcal{P} = (T(TT)) - \frac{1}{4}\partial^2(TT) + \frac{1}{56}\partial^4 T$.

We use the notation $(AB)(z)$ representing the normal ordering of two operators $A(z)$ and $B(z)$. Note that at level 6, $\mathcal{P}(z)$ and $\mathcal{Q}(z)$ are not orthogonal. After using the Gram-Schmidt orthogonalization process, we get the orthogonalized operators $\mathcal{Q}(z)$ and $\mathcal{R}(z)$.

Normalization factor α_k

Firstly one define the state $|k\rangle \equiv \phi_k(0,0)|0\rangle$, with $|0\rangle$ being the vacuum state of the CFT on C, and then

$$\alpha_k = \langle k|k\rangle. \tag{4.1}$$

For example, for the operator $\mathcal{O}(z)$ we have

$$|\mathcal{O}\rangle = \left(L_{-2}L_{-2} - \frac{3}{5}L_{-4}\right)|0\rangle, \tag{4.2}$$

and then

$$\alpha_{\mathcal{O}} = \frac{c(5c+22)}{10}. \tag{4.3}$$

Similarly, for other quasiprimary operators, their normalization factors are respectively

$$\alpha_1 = 1, \quad \alpha_T = \frac{c}{2}, \quad \alpha_{\mathcal{Q}} = \frac{4c(70c+29)}{63},$$
$$\alpha_{\mathcal{R}} = \frac{3c(2c-1)(5c+22)(7c+68)}{4(70c+29)}. \tag{4.4}$$

Quasiprimaries in CFT$_1$

There are also the antiholomorphic quasiprimary operators $\bar{T}$, $\bar{\mathcal{O}}$, $\bar{\mathcal{Q}}$ and $\bar{\mathcal{R}}$, as well as the quasiprimary operators with mixing holomorphic and antiholomorphic parts. Explicitly, at each level $L_0 + \bar{L}_0$, we have

- At level 0, it is 1.
- At level 2, they are T and $\bar{T}$.
- At level 4, they are $\mathcal{O}$, $\bar{\mathcal{O}}$ and $T\bar{T}$.
- At level 6, they are $\mathcal{Q}$, $\mathcal{R}$, $\bar{\mathcal{Q}}$, $\bar{\mathcal{R}}$, $T\bar{\mathcal{O}}$ and $\bar{T}\mathcal{O}$.

Note that here the quasiprimary operators are just trivial multiplications of the holomorphic and antiholomorphic parts, because that the OPE of T and $\bar{T}$ has no singular terms.

Quasiprimaries in CFT$_n$

The quasiprimary operators are listed as below.

L_0	quasiprimary operators	degeneracies	#
0	1	1	1
2	$T(z_j)$	n	n
4	$T(z_{j_1})T(z_{j_2})$ with $j_1 < j_2$	$\frac{n(n-1)}{2}$	$\frac{n(n+1)}{2}$
	$\mathcal{O}(z_j)$	n	
5	$\mathcal{S}_{j_1 j_2}(z)$ with $j_1 < j_2$	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$
	$T(z_{j_1})T(z_{j_2})T(z_{j_3})$ with $j_1 < j_2 < j_3$	$\frac{n(n-1)(n-2)}{6}$	
	$T(z_{j_1})\mathcal{O}(z_{j_2})$ with $j_1 \neq j_2$	$n(n-1)$	
6	$\mathcal{U}_{j_1 j_2}(z)$ with $j_1 < j_2$	$\frac{n(n-1)}{2}$	$\frac{n(n+1)(n+5)}{6}$
	$\mathcal{Q}(z_j)$	n	
	$\mathcal{R}(z_j)$	n	
...	...	...	...

Note that the j's listed above vary as $0 \leq j \leq n-1$, and also the operators

$$\mathcal{S}_{j_1 j_2}(z) = T(z_{j_1}) i\partial T(z_{j_2}) - i\partial T(z_{j_1}) T(z_{j_2}),$$
$$\mathcal{U}_{j_1 j_2}(z) = \frac{5}{9}\partial T(z_{j_1})\partial T(z_{j_2}) - \frac{2}{9}\partial^2 T(z_{j_1}) T(z_{j_2}) - \frac{2}{9} T(z_{j_1})\partial^2 T(z_{j_2})$$

can not be factorized into the operators at different copies.
The coefficients α_K for these operators could be calculated easily

$$\alpha_{TT} = \frac{c^2}{4}, \quad \alpha_{\mathcal{S}} = 2c^2, \quad \alpha_{TTT} = \frac{c^3}{8},$$
$$\alpha_{T\mathcal{O}} = \frac{c^2(5c+22)}{20}, \quad \alpha_{\mathcal{U}} = \frac{20c^2}{9}. \tag{4.5}$$

The coefficient d_K

To compute d_K we consider the multivalued transformation

$$z \to f(z) = \left(\frac{z-\ell}{z}\right)^{1/n}, \tag{4.6}$$

which maps the Riemann surface $\mathcal{R}_{n,1}$ to the complex plane C. With some efforts, we can get d_K for various operators listed above,

$$d_1 = 1, \quad d_T = \frac{n^2-1}{12n^2}, \quad d_{TT}^{j_1j_2} = \frac{1}{8n^4 c}\frac{1}{s_{j_1j_2}^4} + \frac{(n^2-1)^2}{144n^2},$$

$$d_{\mathcal{O}} = \frac{(n^2-1)^2}{288n^4}, \quad d_{\mathcal{S}}^{j_1j_2} = \frac{1}{16n^5 c}\frac{c_{j_1j_2}}{s_{j_1j_2}^5},$$

$$d_{TTT}^{j_1j_2j_3} = -\frac{1}{8n^6c^2}\frac{1}{s_{j_1j_2}^2 s_{j_2j_3}^2 s_{j_1j_3}^2} + \frac{n^2-1}{96n^6c}\left(\frac{1}{s_{j_1j_2}^4}+\frac{1}{s_{j_2j_3}^4}+\frac{1}{s_{j_1j_3}^4}\right) + \frac{(n^2-1)^3}{1728n^6},$$

$$d_{T\mathcal{O}}^{j_1j_2} = \frac{n^2-1}{96n^6c}\frac{1}{s_{j_1j_2}^4} + \frac{(n^2-1)^3}{3456n^6}, \quad d_{\mathcal{Q}} = -\frac{(n^2-1)^2\left(2(35c+61)n^2-93\right)}{5760n^6(70c+29)},$$

Here $s_{j_1j_2} \equiv \sin\frac{\pi(j_1-j_2)}{n}$ and $c_{j_1j_2} \equiv \cos\frac{\pi(j_1-j_2)}{n}$.

Application I: one short interval on cylinder

- We choose the coordinate of the cylinder be z and the subsystem A to be an interval $A = [0, \ell]$ with $\ell \ll L$.
- The Rényi entanglement entropy of A is known exactly P. Calabrese and J. Cardy 0405152

$$S_n = \frac{c}{6}\left(1 + \frac{1}{n}\right) \log\left(\frac{L}{\pi\epsilon} \sin\frac{\pi\ell}{L}\right). \tag{5.1}$$

- From OPE of twist operators

$$\mathrm{Tr}\rho_A^n = \langle\sigma(\ell,\ell)\tilde{\sigma}(0,0)\rangle_L = c_n \ell^{-\frac{c}{6}\left(n-\frac{1}{n}\right)} \sum_K d_K \ell^{h_K+\bar{h}_K} \langle\Phi_K(0,0)\rangle_L,$$

- Due to the translational invariance, the expectation value of one operator on the cylinder $\langle\Phi_K(z,\bar{z})\rangle_L$ must be independent of the coordinates, and so the derivative terms vanish uniformly.

Finite size correction

- The holo. and anti-holo. sectors are decoupled, the computation could be simplified more

$$\mathrm{Tr}\rho_A^n = c_n \ell^{-\frac{c}{6}\left(n-\frac{1}{n}\right)} \left(\sum_K d_K \ell^{h_K} \langle \Phi_K(0) \rangle_L \right)^2,$$

with K being the summation over all the linear independent holomorphic quasiprimary operators.

- In the end, we could find the Rényi entanglement entropy

$$\begin{aligned} S_n &= -\frac{1}{n-1} \log \mathrm{Tr}\rho_A^n \\ &= \frac{c}{6}\left(1+\frac{1}{n}\right)\left(\log\frac{\ell}{\epsilon} - \frac{\pi^2\ell^2}{6L^2} - \frac{\pi^4\ell^4}{180L^4} - \frac{\pi^6\ell^6}{2835L^6} + O\left(\frac{\ell}{L}\right)^8\right) \end{aligned}$$

which matches (5.1) to the order of $O(\ell^6)$.

Application II: Two intervals with small cross ratio

$$
\begin{aligned}
\mathrm{Tr}\rho_A^n &= c_n^2 y^{-\frac{c}{3}\left(n-\frac{1}{n}\right)} \left(\sum_K \alpha_K d_K^2 y^{2h_K} \right. \\
&\left. \sum_{m,p\geq 0} (-)^m \frac{(m+p)!}{m!p!} a_K^m a_K^p C_{2h_K+m+p-1}^{m+p} y^{m+p} \right)^2,
\end{aligned}
$$

with $y^2 = x$.

- With the coefficients d_K obtained before, the computation is straightforward but tedious

Rényi mutual information

- The Rényi mutual information is

$$
\begin{aligned}
I_n &= \frac{c}{3}(1+\frac{1}{n})\log\frac{y}{\epsilon} + \frac{1}{n-1}\log \mathrm{Tr}\rho_A^n, \\
&= I_n^{tree} + I_n^{1-loop} + I_n^{2-loop} + \cdots .
\end{aligned} \tag{5.2}
$$

- Here we have classified the contributions according to the order of the inverse of central charge $\frac{1}{c}$, which in the large c limit corresponds to tree, 1-loop, and 2-loop contributions in the gravity side
- After some highly nontrivial summation...

Useful formulae I

Define

$$f_m(n) = \sum_{j=1}^{n-1} \frac{1}{\left(\sin \frac{\pi j}{n}\right)^{2m}},$$

we need

$$f_1(n) = \frac{n^2-1}{3}, \quad f_2(n) = \frac{(n^2-1)(n^2+11)}{45},$$

$$f_3(n) = \frac{(n^2-1)(2n^4+23n^2+191)}{945},$$

$$f_4(n) = \frac{(n^2-1)(n^2+11)(3n^4+10n^2+227)}{14175},$$

$$f_5(n) = \frac{(n^2-1)(2n^8+35n^6+321n^4+2125n^2+14797)}{93555},$$

$$\sum_{0\le j_1<j_2<j_3\le n-1} \frac{1}{s^2_{j_1j_2}s^2_{j_2j_3}s^2_{j_1j_3}} = \frac{n(n^2-1)(n^2-4)(n^2+47)}{2835},$$

Useful formulae II

$$\sum_{0\le j_1<j_2<j_3\le n-1}\frac{1}{s^4_{j_1j_2}s^4_{j_2j_3}s^4_{j_1j_3}}=\frac{n\left(n^2-1\right)\left(n^2-4\right)}{273648375}\times \left(19n^8+875n^6+22317n^4+505625n^2+5691964\right)$$

$$\sum_{0\le j_1<j_2<j_3\le n-1}\left(\frac{1}{s^4_{j_1j_2}}+\frac{1}{s^4_{j_2j_3}}+\frac{1}{s^4_{j_1j_3}}\right)=\frac{n(n^2-1)(n-2)\left(n^2+11\right)}{90},$$

$$\sum_{0\le j_1<j_2<j_3\le n-1}\left(\frac{1}{s^4_{j_1j_2}}+\frac{1}{s^4_{j_2j_3}}+\frac{1}{s^4_{j_1j_3}}\right)^2=\frac{n(n^2-1)(n-2)\left(n^2+11\right)}{28350}\times \left(3n^4+8n^3+26n^2+152n+531\right)$$

Mutual information: classical part

The tree part, or the so-called classical part, being proportional to the central charge c, is

$$\begin{aligned} I_n^{tree} = & \frac{c(n-1)(n+1)^2x^2}{144n^3} + \frac{c(n-1)(n+1)^2x^3}{144n^3} \\ & + \frac{c(n-1)(n+1)^2\left(1309n^4-2n^2-11\right)x^4}{207360n^7} \\ & + \frac{c(n-1)(n+1)^2\left(589n^4-2n^2-11\right)x^5}{103680n^7} \\ & + \frac{c(n-1)(n+1)^2}{156764160n^{11}} \cdot \\ & \cdot \left(805139n^8-4244n^6-23397n^4-86n^2+188\right)x^6 + O\left(x^7\right) \end{aligned}$$

This matches the result in M. Headrick 1006.0047, T. Hartman 1303.6955, T. Faulkner 1303.7221.

Mutual information: 1-loop correction

The quantum 1-1oop part, being proportional to c^0, is

$$I_n^{1-loop} = \frac{(n+1)\left(n^2+11\right)\left(3n^4+10n^2+227\right)x^4}{3628800n^7}$$
$$+\frac{(n+1)\left(109n^8+1495n^6+11307n^4+81905n^2-8416\right)x^5}{59875200n^9}$$
$$+\frac{(n+1)\left(1444050n^{10}+19112974n^8+140565305n^6+1000527837n\right.}{523069747200n^{11}}$$
$$+O\left(x^7\right),$$

and this matches the result in M. Headrick 1006.0047, T. Barrella 1306.4682.

Mutual information: 2-loop correction

Remarkably there is also the quantum 2-loop contribution, being proportional to $1/c$,

$$I_n^{2-loop} = \frac{(n+1)\left(n^2-4\right)x^6}{70053984000n^{11}c} \cdot \left(19n^8+875n^6+22317n^4+505625n^2+5691964\right) + O\left(x^7\right),$$

This is novel, expected to be the quantum 2-loop contribution in gravity.

Conclusion

- Rényi entropy and its 1-loop quantum correction in the AdS_3 gravity sheds new light on the AdS_3/CFT_2 correspondence
- We developed the short interval expansion of twist operators by considering the derivatives of the quasiprimary operators
- This allowed us to get the subleading contributions of Rényi entropy
- To order 6 in the short interval expansion, we reproduced exactly the classical and 1-loop quantum contributions to the Rényi entropy
- Strong support of holographic computation of EE and RE

Discussion

- Rényi entropy opens a new window to study the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence
- In the case of two disjoint intervals, the Rényi entropy S_2 is just the partition function on a torus with a modular parameter. This partition function corresponds to the 1-loop determinant of physical fluctuations around the thermal AdS space.
- The higher Rényi entropy $S_n, n > 2$ present new challenges and criterion
- What's the CFT dual of quantum AdS_3 gravity? E. Witten 1988, S. Carlip 050302, A. Maloney and E. Witten 0712.0155

Discussion

Our investigations in this work could be extended in several directions.

- First of all, it would be interesting to compute the Rényi entropy of a concrete CFT model, considering the limited knowledge on this issue
- Secondly, it would be interesting to study the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence with other matter coupling. In particular, the Rényi entropy may provide another window to check the minimal model holography in M. Gaberdiel and R. Gupakumar 1207.6697.
- Thirdly, it would be worthwhile to discuss the Rényi entropy in the gravity with higher derivative corrections J. deBoer 1101.5781, L-Y. Hung 1101.5813, BC and J-j. Zhang 1305.6767
- It would be nice to generalize our study to the case with more than two intervals
- It is certainly important to generalize our prescriptions to higher dimensions

Thanks for your attention!